Pergamon

Chaos, Solitons & Fractals, Vol. 4, No. 1, pp. 133-173, 1994
Elsevier Science Ltd

0960-0779/94$6.00 + .00

A Comparative Classification of Complexity Measures

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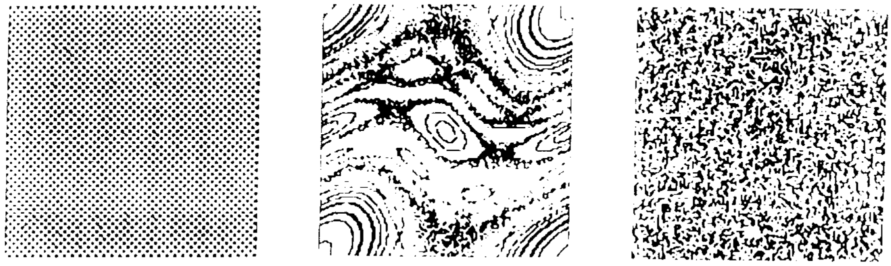
Abstract – A number of different measures of complexity have been described, discussed, and applied to the logistic map. A classification of these measures has been proposed, distinguishing homogeneous and generating partitions in phase space as well as structural and dynamical elements of the considered measure. The specific capabilities of particular measures to detect particular types of behavior of dynamical systems have been investigated and compared with each other.

1. INTRODUCTION

1.1 Complexity

The notion of complexity has been object of numerous and extensive studies since it has become clear that the exact sciences, in particular physics, can no longer afford to disregard the behavior of systems which cannot be treated simply. A simple treatment has always been assumed to be possible either if only few degrees of freedom are involved or if central limit theorems can be applied in case of many degrees of freedom. These assumptions cannot be maintained for nonlinear dynamical systems in general. In such systems, complex (in contrast to simple) behavior can occur with only few degrees of freedom, and central limit theorems are not always applicable.

A very clear and suggestive illustration of a basic issue arising in the context of defining complexity is due to Grassberger [1]. It is reproduced in Figure 1 and it shows three patterns corresponding to a



(a) (b) (c)

Fig. 1. Three patterns used to demonstrate that the pattern that one intuitively would call the most complex (b) is not the one with highest Shannon information (and algorithmic complexity) (c) nor the one with lowest (a). Reproduced from Grassberger [1] with kind permission of the author

different degree of complexity. Figure 1a shows a completely regular, ordered structure, Figure 1c shows a completely random, disordered structure, and Figure 1b represents a mixture of order and disorder, regularity and randomness. From an intuitive point of view, the latter pattern will quite naturally be considered most complex by most people. This intuitive judgment, however, contradicts the historically earliest formal measure of complexity as suggested by Kolmogorov in 1965 [2]. It is called algorithmic complexity and has been further developed by Chaitin [3]. The algorithmic (Kolmogorov–Chaitin) complexity of a pattern is essentially given by the length of the shortest algorithm capable of reproducing the pattern. As such, it would assign highest complexity to Figure 1c instead of 1b. Algorithmic complexity is (in many, but not in all cases) a measure of randomness like Shannon information or dynamical entropy, not of complexity in the intuitively appealing sense.

Presently a bunch of different definitions of complexity and corresponding measures of complexity exist. Excellent overviews considering the state of the art up to 1988 have been given by Grassberger [1] as well as by Lindgren and Nordahl [5]. Main current concepts are due to Hopcroft and Ullman [6], Ziv and Lempel [7], Rissanen [8, 9], Wolfram [10], Yao [11], Bennett [12, 13], Grassberger [1, 4], Koppel and Atlan [14], Lloyd and Pagels [15], Crutchfield and Young [16], Langton [17], Huberman and Hogg [18], Bates and Shepard [19]. While complexity according to references [6] to [11] is more or less a measure of randomness as algorithmic complexity, the remaining approaches represent different attempts toward a definition of complexity as suggested by the intuitive notion sketched above. Within this notion, complexity is not monotonically related to dynamical entropy (or Shannon information) but is a globally convex function of it. Complexity is low for minimum and maximum entropy (information), and it is high at intermediate entropy (information) values.

Several authors studying notions of complexity that are in the spirit of the latter approach have found indications that complexity becomes maximal whenever a system's control parameter passes a value at which the behavior of the system switches between regular and chaotic. It has even been suspected that complexity at those "phase transitions" diverges in the thermodynamic limit. At phase transitions, the information processing (computational) capabilities of a dynamical system are regarded to be very high. In Grassberger's terminology, this corresponds to the capability to master a meaningful task of very high degree of difficulty [4]. From a thermodynamic point of view, a system produces a maximum amount of fluctuations at a phase transition or, respectively, at an instability, if non-equilibrium systems are concerned.

1.2 Basic Concepts

Although there is already a basic distinction between complexity à la Kolmogorov-Chaitin (algorithmic complexity) and complexity according to the intuitive notion presented above, the large and still increasing number of complexity measures calls for a more detailed scheme into which these measures may be categorized. In the present section such a scheme will be introduced. It is explicitly built upon the antinomies of *structural* versus *dynamical* properties of point sets and of *homogeneous* versus *generating* partitions on their support. The basic types of support considered here are phase space and position space.

- *Homogeneous partitions* (P^H) are partitions into cells (boxes, balls, etc.) of identical volume with respect to the Lebesgue-measure. This kind of partition corresponds to a homogeneous concept of space, and it is practically easy to handle. Apart from these, no additional reasons exist that might favor the use of homogeneous partitions compared with any other, arbitrarily chosen inhomogeneous partition. A homogeneous partition implies an appeal of universality, since each measure derived within a corresponding partition is independent of (context–free with respect to) any specific properties of the system concerned.

- *Generating partitions* (P^G) are partitions into cells whose boundaries are generated by the properties (in particular by the dynamics) of the system under investigation. The most important feature of this type of partition is that boundaries between cells are always mapped onto themselves during the evolution of the system. This means that any generating partition is a Markov partition (but not vice versa). The payoff for this advantage is that a generating partition has to be constructed for each system individually and requires knowledge of the dynamical laws governing its evolution. In this sense it turns out to be system–specific, hence context–dependent in contrast to universal.

- *Structural measures* (S) of a system are measures of those of its properties which do not expl[illegible] contain information about its dynamics. The formal basis of a measure for such properties a[illegible] probabilities p_i to find a point in a given cell (state) of the chosen partition. The mathema[illegible] "measure" of a point set, i.e. a probability distribution, is a prototype of such a structural meas[illegible] For example, the set of Renyi dimensions as well as the set of Renyi entropies are structural meas[illegible] in this sense.

- *Dynamical measures* (D) of a system are measures of those of its properties which contain info[illegible] tion about its dynamics. Their formal basis is given by the transition probabilities $p_{i\to j}$ between [illegible] for successive time steps. They consider the dynamical behavior of a system in terms of its tem[illegible] evolution. Practically speaking, measures containing transition probabilities without any [illegible] use of an underlying probability distribution are not known. In this sense, dynamical measu[illegible] complexity appear to be useful only in combination with structural components. Some comple[illegible] measures (for example information gain, mutual information) are defined by transition proba[illegible] ities, which can formally be reduced to state probabilities. These measures will be considered structural measures.

Combining these four separate criteria of classification, four different classes of measures of comple[illegible] can be obtained. Subsequently they are listed together with aspects of their suggested interpretation [illegible] application.

SP^H Structural measures based on a homogeneous partition.

These measures are suggested for use with respect to structural properties of systems as [illegible] are directly and concretely observable in usual (external) position space R^3. They are based [illegible] a homogeneous partition into cells of identical size. This type of partition is relevant, if a p[illegible] set is considered independent of the underlying creating dynamics (e.g. for a given distribu[illegible] of points without knowledge of the dynamics). It can be justified as well by the assumptio[illegible] local homogeneity of space, which enables that applications to small scale systems appear [illegible] reasonable.

SP^G Structural measures based on a generating partition.

As mentioned above, these measures are suitable with respect to the characterization of structu[illegible] properties of systems as they are represented in phase space, but they require knowledge of [illegible] dynamics by which the structure has been generated. Once the dynamics is known, the contextua[illegible] of the generating partition enables a unique definition of its own history. In contrast to the conc[illegible] relevance of structures in external position space, structures in phase space can be regarded abstract structures. Phase space coordinates are different from position space coordinates in a formal and in a conceptual sense.

DP^H Dynamical measures based on a homogeneous partition.

These measures refer to dynamical, but abstract properties of a system in terms of a parameter time t which is used in order to define rates of change, e.g., velocity as the temporal change of (external) position. Parameter time t is abstract insofar as it is not object to direct perception in external position space. Any measurement of time t is based on an observation of concretely perceivable positions, e.g., of the pointers of a clock. In this sense, t can be understood as an external, but abstract concept of time. It is crucial for all types of unitary dynamical laws, and corresponds to the point of view of temporal reversibility.

DP^G Dynamical measures based on a generating partition.

In contrast, concrete dynamical properties of a system can be characterized by the concept of an internal time τ. In an illustrative manner, notions like age, memory, and related phenomena of decaying correlations between quantities defined locally in phase space fall into this category. For instance, intrinsically instable systems with positive Kolmogorov-Sinai-entropy K possess an "intrinsic" correlation time given by K^{-1}. (This does not contradict the existence of longer, non-"intrinsic" correlation times, e.g., due to memory effects.) Direct operational access to τ seems to be possible only from the interior perspective of the system itself. In this perspective, internal time is responsible for all types of phenomenological arrows of time, hence emphasizes the point of view of temporal irreversibility.

The preceding characterization entails the antinomies of internality and externality (endo/exo) as well as of abstraction and concreteness, the latter one coinciding with that of the descriptive level of models and the observational level of systems. For more details on a formal treatment of these antinomies and their relationships with each other, we refer to a more detailed discussion given elsewhere [20, 21].

1.3 Organization of the Article

Section 2 of the present paper is devoted to the notions of structural and dynamical measures of complexity, to their definition, and to the assignment of specific existing complexity measures to both of these categories. For instance, concepts like algorithmic complexity, generalized information, and information gain will be classified as structural measures. In contrast, concepts based on fluctuation of information gain, as well as on machine complexity belong to dynamical measures. An inexhaustive, but hopefully representative number of corresponding examples will be discussed. The investigated measures of complexity together with their behavior in the limiting cases of periodic and stochastic behavior will finally be summarized.

Section 3 applies the measures as treated in Section 2 to the example of the logistic map. In particular the behavior of those measures is studied in the periodic regime, at the onset of chaos (accumulation point), at band merging, in the period-3 window, and at fully developed chaos. It is shown that different measures are indicators of different quality for different kinds of dynamical behavior. A summarizing comparison of these differences is given.

Section 4 concludes the paper and relates its content to the ongoing and controversial discussion of meaning. Together with the significance of context, the notion of meaning might serve to understand complexity as an extension of the simplifying framework of purely universal (context-free) principles and purely syntactic (semantic-free) descriptions.

2. QUANTIFICATION OF COMPLEXITY BY STRUCTURAL AND DYNAMICAL MEASURES

2.1 General Definitions

In this paper we consider *one-dimensional dynamical systems in discrete time*, given by a map F from a closed interval $\mathbf{A}$ onto itself:

$$F: \quad \mathbf{A} \rightarrow \mathbf{A}, x \mapsto F_r(x) \tag{1}$$

$x \in \mathbf{A}$ is called state of the system, the range of $r \in R$ represents the parameter space of the system. The corresponding set of dynamical systems is characterized by discreteness in time but continuity in state space. In case of dynamical systems with continuous time as, e.g., any kind of differential equations, a transformation into time discrete maps is possible by various methods (Poincaré sections, stroboscopic maps [22, 23]).

From a temporally discrete, spatially continuous map a *symbolic dynamical system* can be generated by an additional discretization of the state space $\mathbf{A}$ [24]. If the state space $\mathbf{A}$ of a dynamical system is divided into N cells A_i that are non-empty with respect to the Lebesgue measure of the system, then the collection of all cells is called a ***partition*** $P = \{A_i\}_{i=1}^{N}$ if the A_i are mutually disjoint and the union of A_i reproduces the state space: $\bigcup_{i=1}^{N} A_i = \mathbf{A}$.

By labeling each element of the partition $P = \{A_i\}_{i=1}^{N}$ with a symbol a_i, the time evolution of the dynamical system (i.e., its discretized trajectory) can be expressed by a symbol sequence $S = s_0 s_1 s_2 \ldots$. This sequence is constructed such that after each time step i the state of the system x_i is assigned to the corresponding symbol s_i. This symbol is determined by the cell A_j that is met by the trajectory at time i. The set of all possible symbols $s_i \in \{a_0, a_1, \ldots a_{N-1}\}$ is called an alphabet of cardinality N. The resulting *symbolic dynamical system* is defined as

$$\Sigma_F \to \Sigma_F, \quad S \mapsto \hat{\sigma}_F(S) = S' \tag{2}$$

such that each symbol in the sequence S satisfies the condition $s_{i+1} = s_i' = \hat{\sigma}_F(s_i)$.
Σ_F is the space of all admissible symbol sequences. Admissible sequences are sequences that are induced by the dynamics of the system F for all initial states $x_0 \in \mathbf{A}$ at time step $i = 0$. The operator $\hat{\sigma}_F$ is called the *shift operator on* Σ_F and describes the dynamics generated by F in the space Σ_F of symbol sequences. The length L of a symbol sequence S is defined by $S = \{s_i\}_{i=0}^{L-1}$. In principle, the theory of symbolic dynamical systems deals with sequences of infinite length ($L = \infty$). For practical purposes, however, L is often regarded as a finite number.
The symbolic dynamical system (Eq.(2)) is constructed in a way that leaves it topologically equivalent to Eq.(1) [24]. This implies a well-defined assignment of trajectories to symbol sequences that represent the topological properties of the underlying dynamical system F (e.g., the number of periodic points of the system) faithfully.

As indicated in section 1.2, different kinds of partition can be utilized in order to discretize the state space $\mathbf{A}$.

1) A homogeneous, context-free partition P^H into cells A_i^H of identical size: $P^H = \{A_i^H\}_{i=1}^{N^H}$, where N^H is the number of states (cells) in the state space $\mathbf{A}$.

2a) A generating partition P^G, or P_n^G, generated by the dynamics of the system: $P^G = \{A_i^G\}_{i=1}^{N^G}$. N^G is the number of cells (states) in $\mathbf{A}$. If necessary to avoid misunderstanding, an index n will be added to P_n^G, N_n^G, or $A_{i,n}^G$, respectively, characterizing the number of time steps considered to generate the partition. This index n is also called the *order of refinement* of the generating partition. More details will be given later. A generating partition P^G creates arbitrarily small cells as the order n of refinement goes to infinity. Given trajectories can be assigned to corresponding symbol sequences in a well-defined manner [25].

2b) Based on some alphabet $\{a_0, a_1, \ldots, a_{N_1^G-1}\}$, the set P_n^{G*} of all possible subsequences (words) of length n out of a symbol sequence S represents the total set of "trajectories" with respect to all possible initial states over a time interval of n time steps. This set can be understood as an *cylinder-induced partition* $P_n^{G*} = \{A_{i,n}^W\}_{i=1}^{N_n^W}$, where $A_{i,n}^W \in \{s_k s_{k+1} \ldots s_{k+n-1} | k = 0, 1, \ldots, L\}$ is called a word of length n. $A_{i,n}^W$ defines the partition corresponding to P_n^G in the space of symbol sequences Σ_F. For a generating n-cylinder-induced partition P_n^{G*} the number of admissible words $A_{i,n}^W$ in the sequence S increases with n, such that there is a well-defined assignment of trajectories (considered over n time steps) to words $A_{i,n}^W$.

For a given alphabet and a given order n of refinement, the number N_n^W of admissible words of length n is equal to the number of cells $A_{i,n}^G$ in **A** with non vanishing natural measure. This holds, too, for the natural measure on P^{G*} and P^G. In this sense both partitions, P^{G*} and P^G, are equivalent, but certain complexity measures require P^{G*} for their construction.

In general, different partitioning procedures as defined above produce different partitions $P^H \neq P^G$. However, there are exceptional cases in which the resulting partitions are identical. For instance one obtains $P^H = P^G$ for the symmetric tent map [26].

The generating partition P^G is a *Markov partition*, since it has the fundamental property:

$$F_r(A_k^G) \cap A_j^G \neq \{\} \quad \Longrightarrow \quad F_r(A_k^G) \supset A_j^G \quad \forall \quad j, k = 1, \ldots, N^G$$

This implies that boundaries between cells are mapped onto themselves: they are kept invariant during the dynamical evolution of a system, if the cells are constructed by a generating partition [27]. In case of a homogeneous partition P^H, boundaries between cells are in general not invariant with respect to the dynamics of the system.

To analyze the dynamics generated by F_r, we now define some probabilistic quantities used to characterize given states $A_i, A_j (i, j = 1, \ldots, N)$. Note that A_i can refer to A_i^H, A_i^G, or A_i^W.

1. The state probability p_i is the probability that a trajectory on state space **A** visits box A_i. It is defined by the natural measure μ: $p_i := \mu(A_i), i = 1, \ldots, N$.
   In case of P^{G*} this is equivalent to the probability that a given word A_i^W of length n appears in the symbol sequence S.

2. Using the joint probability p_{ij} that cells A_i and A_j are visited by the system in two successive time steps we define the transition probabilities $p_{i\rightarrow j}$ and $p_{i\leftarrow j}$ $(i, j = 1, \ldots, N)$:
   $p_{i\rightarrow j} := \frac{p_{ij}}{p_i}$ is the conditional probability for the transition from a given state (cell) A_i to a successive state A_j.

$p_{i \leftarrow j} := \frac{p_{ij}}{p_j}$ is the conditional probability for the transition to a given state A_j from its predecessor A_i.

The probabilities $p_{i \to j}$ and $p_{i \leftarrow j}$ can be repesented using a transition matrix, whose rows sum up to one.

In case of P_n^{G*}, both transition probabilities are defined for sucessive time steps if the states are represented by $A_{i,n}^W = s_k s_{k+1} \dots s_{k+n-1}$ and $A_{j,n}^W = s_{k+1} s_{k+2} \dots s_{k+n}$ for all possible k, $k \in \{0, \dots, L-n\}$, of the symbol sequence S.

Under the assumption that the various kinds of possible dynamical behavior of a system (periodic, chaotic, uniformly stochastic, and doubly stochastic) can be resolved in the coarse-grained partition of state space, they can be characterized in terms of these probabilities.

- *Periodic behavior of period $\wp$, $\wp \leq N$:*

  Without restrictions upon generality let $p_i \neq 0 \quad \forall \quad i = 1, \dots, \wp$ for the following. In case of periodic behavior with period $\wp$, $\wp \leq N$ and $i, j \in \{1, \dots, \wp\}$, we have $p_i = 1/\wp$ for the state probabilities and $p_{i \to j} = 1$ for the transition probabilities. All remaining state and transition probabilities for $i, j \in \{\{1, \dots, N\} \times \{1, \dots, N\} | p_{i \to j} \neq 1\}$ have to vanish. Periodic behavior is undetectable if the number of states N is smaller than the period $\wp$ of the system.

  As an alternative criterion, a *periodic point x of period $\wp$* satisfies the condition $F^{\wp}(x) = x$, where $F^0(x) = x, F^{\wp}(x) = F(F^{\wp-1}(x))$. Analogously a symbol sequence of period $\wp$ satisfies $s_{i+\wp} = s_i$ $\forall \quad i = 0, \dots, L - \wp$. The least positive $\wp$ for which $F^{\wp}(x) = x$ or $s_{i+\wp} = s_i$ holds is called the *prime period of x or S.*

- *Uniformly stochastic behavior:*

  Uniformly stochastic behavior is characterized by an equi–distribution of the probabilities $p_i = 1/N$ $(i \in \{1, \dots, N\})$ of all possible states on the space $\mathbf{A}$ with respect to some arbitrary, but fixed refinement of the partition. Transition probabilities are not considered in this context.

- *Doubly stochastic behavior:*

  For *doubly stochastic behavior* [27] we require $p_i = 1/N$ as well as $p_{i \to j} = 1/N'$ with $N' \leq N$ for all $i, j \in \{1, \dots, N\}$. N is again the number of states on $\mathbf{A}$. N' is the number of transitions with nonvanishing transition probabilities for each cell A_i, $\quad i = 1, \dots, N$. (Note that $N' = N_1$, respectively $N' = 2$ for a binary alphabet.)

  Stochasticity in this sense is characterized by uniformity of state probabilities *and* transition probabilities, thus motivating the notion of *doubly* stochastic behavior.

  For $N = N'$ doubly stochastic behavior is equivalent to a coin-tossing process, which implies *total* *independence* of successive states: $p_{ij} = p_i p_j \quad \forall \quad i, j = 1, \dots, N$. In this case, all possible states are admissible. White noise is an example for such a completely random process, whereas colored noise is not doubly stochastic in general.

As remarked in the introduction the definitions of some measures of type SP^{H}, SP^{G}, SP^{C} (information gain, mutual information, and effective measure complexity) depend on transition probabilities $p_{i\to j}$, which can be reduced to state probabilities. In this case, an equi-distribution of transition probabilities due to doubly stochastic behavior implies an equi-distribution of state probabilities p_i at time step n and of p_j at time step $n+1$. Both aspects can be used to detect doubly stochastic behavior.

- *Chaotic behavior:*

  Chaotic behavior does not imply any restrictions upon the probabilistic measures discussed so far. Any distribution of p_i and $p_{i\to j}$ can occur as long as the general normalization criteria for probabilities are satisfied: $\sum_{i=1}^{N} p_i = 1$ and $\sum_{j=1}^{N} p_{i\to j} = 1 - \delta_{p_i,0}$ where $\delta_{\cdot,\cdot}$ is Kronecker's delta.

2.2 Structural Complexity Measures: SP

Based on the framework of the formalism of symbolic dynamical systems this subsection describes a number of selected measures to characterize the complexity of a system in terms of its structural properties. This is to say that spatial properties will be considered whereas the explicit dynamical behavior will be disregarded. Depending on the investigated problem, either structures in position space or in phase space may be relevant. In the first case, the notion of concrete structures is appropriate, whereas the second case obviously deserves to be denoted as abstract.

2.2.1 Algorithmic Complexity

The historically earliest and probably most popular measure of complexity has been introduced by Kolmogorov in 1965 [2]. It is called ***algorithmic complexity***, and it is defined as the number of bits of the shortest algorithm (e.g., computer program) which is capable of reproducing a given symbol sequence. A practical realization of this theoretical approach has been proposed by Ziv and Lempel [illegible]. The Ziv-Lempel-algorithm is often applied as a convenient method to compress data strings [28]. The procedure is to divide the sequence S into subsequences (words) $A^W_{1,n_1}, A^W_{2,n_2}, \dots$ such that $A^W_{1,1} = s_0$, $A^W_{i+1,n_{i+1}} = s_k \dots s_{k+n_{i+1}-1}$ is the shortest word that cannot be copied from the subsequence $S' = s_0 s_1 \dots s_{k+n_{i+1}-2}$.

To give an example, the sequence $S = 1101001111010010\ldots$ of length L splits into $(1)(10)(100)(111)(1010010)\ldots$, providing a number $c(L) = 5$ of resulting words A^W_{i,n_i} of different length n_i. For a symbol sequence consisting exclusively of totally independent symbols, $c(L)$ takes its maximum value according to the relation (logarithms are always binary logarithms):

$$\lim_{L\to\infty}\left(c(L) - \frac{L}{\log L}\log N_1^G\right) = 0 \tag{4}$$

Using this maximum as a normalization factor, the algorithmic complexity C_a is given by:

$$C_a = \lim_{L\to\infty} c(L)\frac{\log L}{L\log N_1^G} \tag{5}$$

where N_1^G is the cardinality of the underlying alphabet.

For periodic behavior with period $\wp$, the relation $c(L) \leq \wp$ provides vanishing complexity C_a. In case of a sequence of totally independent symbols, Eq.(5) leads to $C_a = 1$. Between these limiting cases, algorithmic complexity may take values in the range $0 \leq C_a \leq 1$.

Algorithmic complexity C_a is a non-probabilistic measure. For this reason, it cannot directly be categorized into the scheme introduced above. Nevertheless it may be assigned to the class SP^G of complexity measures, since the number of words $c(L)$ within a symbol sequence is primarily a structural, and definitely not a dynamical system property. Moreover, the symbol sequence is generated by an alphabet based on P^G, which suggests to classify C_a as SP^G-measure. Under the aspect that words of different length n are considered, C_a can be classified as SP^{G*}-measure, as well. The fact that C_a approaches the K-S-entropy $K^{(1)}$ for an infinite sequence [7] confirms this assignment.

2.2.2 Generalized Informations and Related Complexity Measures

Algorithmic complexity shares basic properties with information measures à la Shannon [29]. These measures can very elegantly be captured by a formalism introduced by Renyi [30]. This framework uses the concept of a *generalized information* $I^{(q)}$ on some partition P as it is defined by:

$$I^{(q)} = \frac{1}{1-q}\log\sum_{i=1}^{N} p_i^q \tag{6}$$

For $q \to 1$ the generalized information of order one is given by:

$$I^{(1)} = \lim_{q\to 1} I^{(q)} = -\sum_{i=1}^{N} p_i \log p_i \tag{7}$$

where N is the number of cells A_i, $i = 1,\ldots,N$, for a given partition P^H, P^G, P^{G*}, and $p_i = \mu(A_i)$.

Some important properties of $I^{(q)}$ are:

- $I^{(q)}$ is a monotonically decreasing function of q
- In case of periodic behavior ($p_i = 1/\wp$, $\wp$ prime period, and $\wp \leq N$) one gets: $I^{(q)} = \log \wp$ [illegible] a given period $\wp$ is resolved in a given partition, then the generalized information $I^{(q)}$ is independent of its order q.
- In the uniformly stochastic case ($\wp = N$), Eq.(6) leads to $I^{(q)} = \log N \;\; \forall \; q \in R$. Hence [illegible] diverges for arbitrarily fine partitions ($N \to \infty$).
- On the basis of the concept of generalized informations it is impossible to discriminate [illegible] stochastic from doubly stochastic behavior, since $I^{(q)}$ does neither depend on transition probabilities nor on state probabilities (p_i, p_{ij}) for successive refinements ($n, n+1$).

Applications of generalized information have been studied with respect to partitions of type P^H, [illegible] and P^{G*}. In general, $I^{(q)}$ depends on the particular choice of the partition and its respective refinements. In the following, complexity measures will be discussed that are based on P^H (case A) and on P^G (case B). A) Focussing on homogeneous partitions P^H one can specify a number of additional quantities for characterizing the complexity of a system in terms of its structural properties.

The ***generalized dimensions*** $D^{(q)}$ [31, 32], measures of type SP^H, are defined as the scaling exponents of the generalized information of order q with respect to the size $\varepsilon := 1/N^H$ of cells A_i^H, $i = 1, \ldots, N^H$ of the partition:

$$D^{(q)} = -\lim_{\varepsilon \to 0} \frac{I_\varepsilon^{(q)}}{\log \varepsilon}$$

In contrast to $I^{(q)}$, the dimensions $D^{(q)}$ offer sensitive discrimination of periodic from uniformly stochastic behavior because they are defined for an infinitesimally fine partition ($\varepsilon \to 0$).

- For periodic behavior in case of discrete 1-dimensional maps, the dimension of order q is always given by:

  $$D^{(q)} = -\lim_{\varepsilon \to 0} \frac{\log \wp}{\log \varepsilon} = 0$$

  $D^{(q)}$ is independent of the order q. The speed of convergence for a given refinement (ε) depends on $\log \wp$, i.e. on the period.

- In the uniformly stochastic case $D^{(q)}$ is given by:

  $$D^{(q)} = -\lim_{\varepsilon \to 0} \frac{\log N_\varepsilon}{\log \varepsilon} = 1$$

  for 1-dimensional maps. As in the periodic case, it is constant as a function of q.

B) The second alternative is a generating partition P_n^G or P_n^{G*}, respectively, which provides *generalized entropies* $K^{(q)}$ [33]. Therefore the entropies $K^{(q)}$ represent a structural measure (in an abstract sense), although they are usually (and correctly) characterized as dynamical invariants of dynamical systems. For a given initial uncertainty in state space, $K^{(q)}$ measures the temporal spread of the distribution of admissible trajectories along the attractor. It can be interpreted as a rate of internal information production of the system [34, 35], corresponding to the rate of information loss of an external observer.

$$K^{(q)} = \lim_{n\to\infty} \frac{I_n^{(q)}}{n} \tag{9}$$

$I_n^{(q)}$ is the information of a dynamical system F on a refined partition P_n^G (or P_n^{G*}) of order n. Since P^G and P^{G*} are *generating partitions*, $K^{(q)}$ is the supremum $\sup\limits_P \frac{I_n^{(q)}}{n}$ with respect to all possible partitions P. Then $K^{(q)}$ is the *dynamical entropy of order* q. For $q = 1$, one obtains the Kolmogorov-Sinai entropy, a fundamental invariant of dynamical systems.

The partitions P_n^G and P_n^{G*} are equivalent for given n, since the natural measures of corresponding states $A_{i,n}^G \in P_n^G$ and $A_{i,n}^W \in P_n^{G*}$ are identical ($\mu(A_i^G) = \mu(A_i^W) = p_i$ for all $i = 1, \ldots, N$). As a consequence, determination of $K^{(q)}$ on the basis of P^G and on the basis of P^{G*} provides identical results.

Similar to the generalized dimensions in homogenous partitions, the generalized entropies are given by:

- $K^{(q)} = \lim\limits_{n\to\infty} \frac{\log p}{n} = 0$ in the periodic case.
- $K^{(q)} = \lim\limits_{n\to\infty} \frac{\log N_n}{n}$ for an equi-distribution of state probabilities for all n ($n \to \infty$, and with $N_n = N_n^G$, respectively $N_n = N_n^W$). Thus if the limiting value exists, doubly stochastic behavior is marked.

  In case of (for instance) P_n^{G*} all possible words of length n are admissible, which leads to $N_n = (N_1)^n$ and therefore $K^{(q)} = \log N_1$. (Consider, e.g., a 2 : 1-map. Then the number of cells (states) grows with $N_n = 2^n$, which provides $K^{(q)} = 1$.)

There is a unique relation between $D^{(q)}$ and the $f(\alpha)$- spectrum [36] as well as between $K^{(q)}$ and the $g(\Lambda)$-spectrum [37]. It is therefore not necessary to discuss these spectra (which are often used for an appropriate characterization of multifractals) in detail here.

2.2.3 Local Slopes and Related Complexity Measures

Dynamical entropies $K^{(q)}$ as defined above can be considered as asymptotic slopes of $I_n^{(q)}$ as a function of n. Numerically, this limit is most easily accessible from the "local" slopes [38, 1]:

$$h_n^{(q)} := I_{n+1}^{(q)} - I_n^{(q)}, \qquad h_0^{(q)} := I_1^{(q)} \tag{10}$$

They specify the amount of additional information needed to predict the successive state at time $n+1$ given the state at time step n. For a generating partition P_n^G, P_n^{G*} the local slopes $h_n^{(q)}$ approximate the generalized entropy such that $K^{(q)} = \lim_{n\to\infty} h_n^{(q)}$. For Markov processes of order ν (sometimes denoted as memory ν) the asymptotic value is reached for $n = \nu$ [39].

Measures as information gain (case A), mutual information (case B) and effective measure complexity (case C) are related to these local slopes. They are all considered as structural measures because they depend either on state probabilities (local slopes) exclusively, or on transition probabilities, which are reducable to state probabilities via local slopes.

The local slopes $h_n^{(q)}$ in Eq.(10) are derived from the distribution of state probabilities at two *successive* time steps n and $n+1$. Hence, the dynamical aspect of transition probabilities is replaced by the difference between structural measures at different instants of time.

In principle these measures are applicable to any partition P^H, P^G, P^{G*}, and therefore classified as SP measures in general, i.e. without a priori specification of a particular partition. However, any dynamical measure based on transition probabilities should be defined on P^G, since the dynamics of the system is uniquely related to generating partitions only. In this sense measures related to homogeneous partitions reflect a more or less abstract point of view.

A) An important way to characterize the complexity of dynamical systems is based on the *information gain* G_{ij} (often called Kullback information [40]). It represents the information required to select a state A_j if its preceding state A_i is given.

$$G_{ij} = -\log p_{i\to j} \tag{11}$$

The *mean information gain* $< G >$, i.e. the average of the information gain over all possible transitions $i \to j$, is defined as:

$$< G >= \sum_{i,j=1}^{N} p_{ij} G_{ij} = -\sum_{i,j} p_{ij} \log p_{i\to j} \tag{12}$$

The dependence of $< G >$ on transition probabilities can be reduced to $< G >= -\sum_{i,j} p_{ij} \log p_{ij} + \sum_i p_i \log p_i$. If the second term $\sum_i p_i \log p_i$ corresponds to the information $-I_n^{(1)}$ at time n then the first term describes the information $I_{n+1}^{(1)}$ at time $n+1$. Thus, on a partition of refinement n, Eq.(12) leads to

$$< G >= I_{n+1}^{(1)} - I_n^{(1)} = h_n^{(1)} \tag{13}$$

- For periodic behavior there are strictly determined transitions ($p_{i\to j} \in \{0,1\} \quad \forall \quad i,j = 1,\ldots,N$) and the mean information gain vanishes for all periods of order $\wp$. Used as a measure of complexity $< G >$ is therefore only capable to discriminate periodic from non-periodic behavior. It does not provide information about the period $\wp$ itself.

- $< G >$ is defined for $I^{(1)}$ at two successive time steps $n, n+1$ (Eq. 13). In order to characterize uniformly stochastic behavior, it is appropriate to consider $I^{(1)}$ itself.

- In case of doubly stochastic behavior ($p_i = p_j = 1/N, p_{i\to j} = 1/N'$) one has $< G >= \log \frac{N_{n+1}}{N_n} = \log N'$. For a totally random, statistically independent process for which $N' = N$, this implies $< G >= \log N$.

B) The interdependence of two different states A_i, A_j can be quantified by the measure of *mutual information*:

$$M_{ij} = \log \frac{p_{ij}}{p_i p_j} \tag{14}$$

In stochastically independent cases $p_{ij} = p_i p_j$, one has $M_{ij} = 0$ and the two states A_i, A_j can be considered to be mutually independent in the sense that information about one of them does not depend on information about the other one.

The *mean mutual information* is defined by:

$$< M >= \sum_{i,j=1}^{N} p_{ij} M_{ij} = \sum_{i,j=1}^{N} p_{ij} \log \frac{p_{ij}}{p_i p_j} \tag{15}$$

which provides, using Eqs.(7),(12),(13):

$$< M >= I_n^{(1)} - < G >= 2I_n^{(1)} - I_{n+1}^{(1)} \tag{16}$$

Mutual information $< M >$ is therefore categorized as SP^G-measure (or SP^{G*}-measure, respectively).

- In periodic cases where $< G >= 0$, Eq.(16) yields $< M >= I_n^{(1)} = \log \wp$. Hence mutual information is capable of distinguishing between different periods $\wp$.

- Evaluating Eq.(16) for the doubly stochastic case where $< G >= \log N'$, the mean mutual information is obtained by: $< M >= \log \frac{N}{N'} \geq 0$.
  As a consequence, $N = N'$ implies $< M >= 0$. This applies if the corresponding process is statistically independent (like coin-tossing).

C) Using the concept of "local" slopes for $q = 1$, Grassberger has introduced the *effective measure complexity EMC* [1].

$$
\begin{aligned}
EMC &= \sum_{n=1}^{\infty} n(h_{n-1}^{(1)} - h_n^{(1)}) && (17)\\
&= \sum_{n=0}^{\infty} (h_n^{(1)} - K^{(1)}) && (18)\\
&= \lim_{n\to\infty} (I_n^{(1)} - n \cdot K^{(1)}) && (19)
\end{aligned}
$$

EMC describes the behavior of the local difference $h_n^{(1)}$ if it converges toward the dynamical entrop[illegible] [illegible] the dynamical system. (The convergence of EMC does not always follow from converging loca[illegible] [illegible]ut depends on their precise convergence behavior [illegible].) Small (large) EMC corresponds to fast [illegible] [illegible]onvergence. EMC is based [illegible]n the concept of local slopes and therefore represents a measure [illegible] [illegible] P^G (or SP^{G*}, respectively). It can also be written as an average Kullback information, for insta[illegible] [illegible]rms of conditional probabilities, as demonstrated in [illegible].

- Consider periodic behavior of period $\wp$, which implies $K^{(1)}$ [illegible] and $I^{(\cdot)}$ [illegible] $\log \wp$. Then [illegible] yields:

  $$EMC = \lim_{n \to \infty} I_n^{(1)} = \log \wp$$

  For a constant symbol sequence ($\wp = 1$), EMC vanishes

- For doubly stochastic behavior one has $I_n^{(1)} = \log N_1$ for all n, where n indicates the number [illegible] steps considered. Since in a given alphabet of cardinality N_1 all admissible words of length n [illegible] in the sequence S, the relation $\frac{N_{n+1}}{N_n} = \frac{N_n}{N_{n-1}} = N$, respectively N_n [illegible] $(N_1)^n$, holds. Consequen[illegible] EMC according to Eq.(17) can be reformulated as:

  $$EMC = \sum_{n=1}^{\infty} n \log \frac{N_n^2}{N_{n-1} N_{n+1}} = 0$$

2.3 Dynamical Complexity Measures: DP

Measures considered in this section are dynamical measures in the sense that they depend on both [illegible] [illegible]sition probabilities and state probabilities. (In contrast to measures treated in section 2.2.3, they can[illegible] be reduced to structural measures by expressing transition probabilities in terms of state probabiliti[illegible] different instants.) They are in principle applicable with respect to any of the presented types of partiti[illegible] P^H, P^G, P^{G*}. Therefore they qualify as DP-measures in general. However, analogous to SP^H measu[illegible] related to local slopes, DP^H-measures reflect an abstract point of view: they entail transition probab[illegible] [illegible]ties between cells that are independent of the concrete dynamics of the system. Generating partitio[illegible] P^G take this concrete dynamics into account, and should therefore be preferred whenever dynami[illegible] complexity measures are used.

2.3.1 Fluctuation Complexity

Analogous to mean information gain, defined in section 2.2.3, the *mean information loss* [illegible] L [illegible] is defin[illegible] as the average of *information loss* L_{ij} over all possible transitions $i \leftarrow j$. L_{ij} determines the informatio[illegible] that a system has lost about a preceding state A_i after it has entered the successive state A_j:

$$< L >= \sum_{i,j=1}^{N} p_{ij} L_{ij} = - \sum_{i,j=1}^{N} p_{ij} \log p_{i \leftarrow j} \tag{22}$$

The *net information gain* Γ_{ij} of a system is then expressed by:

$$\Gamma_{ij} = G_{ij} - L_{ij} = \log \frac{p_i}{p_j} \tag{23}$$

Due to the normalization $\sum_j p_{i \to j} = 1$, the *mean net information gain* vanishes: $< \Gamma >= \sum_{i,j} p_{ij} \Gamma_{ij} = 0$. During the evolution of a system, Γ_{ij} may fluctuate about its mean value and therefore may have a non-vanishing mean-square deviation σ_Γ^2. This quantity can be understood as *fluctuation in net information gain*. It has been introduced as a complexity measure by Bates and Shepard [19]:

$$\sigma_\Gamma^2 = < \Gamma^2 > - < \Gamma >^2 \tag{24}$$

$$= \sum_{i,j=1}^{N} p_{ij} (\log \frac{p_i}{p_j})^2 \tag{25}$$

Fluctuation complexity σ_Γ^2 is a dynamical complexity measure since its definition includes both state probabilities and transition probabilities explicitly and irreducibly. It has originally been introduced on a partition P^H, but more adequate use can be made of it, if it is applied to a generating partition P^G (see [41]).

- In case of periodic behavior fluctuation complexity vanishes independent of the prime period $\wp$. In order to detect periodic behavior, sufficient resolution is necessary only with respect to state probabilities, not with respect to transition probabilities (see section 3.3.1).

- σ_Γ^2 depends on transition probabilities $p_{i \to j}$ only if the state probabilities are not equi-distributed. In uniformly stochastic as well as in doubly stochastic cases, i.e., in case of any equi-distribution of state probabilities ($p_i = p_j \quad \forall \quad i, j = 1, \ldots, N$), fluctuation complexity vanishes: $\sigma_\Gamma^2 = 0$. Therefore fluctuation complexity does not distinguish uniformly stochastic from doubly stochastic behavior.

2.3.2 Complexity of ϵ-Machines

The idea to use automata for a definition of complexity goes back to Kolmogorov [2] and led to the concept of algorithmic complexity. It is based on a deterministic automaton and represents, loosely speaking, a measure of randomness. Crutchfield and Young [16] suggested to apply stochastic automata, which they call ϵ-machines. The determination of the complexity C_ϵ of an ϵ-machine can be divided into four main steps.

1. Construction of a tree: A binary tree $T = (V, E, a)$ of length l_1, with a finite set V of vertices, a set $E \subseteq V \times V$ of edges, and an origin $a \in V$ of T, is assigned to a given symbol sequence. This tree describes the probability distribution of P_n^{G*}. A conditional probability $p_e = p_{v_2|v_1}$ is assigned to each edge $e = (v_1, v_2)$, $v_1, v_2 \in V$, of T.

2. Search for equivalent subtrees: From this tree T of length l_1 all subtrees of length l_2 ($l_2 < l_1$) are considered. Within the set of subtrees an equivalence relation (ϵ-similarity) is defined such that any two subtrees are equivalent, iff the difference of the probabilities assigned to their edges e_1, e_2 is smaller than ϵ, $|p_{e_1} - p_{e_2}| < \epsilon$.

3. Construction of random automaton: The condition $|p_{e_1} - p_{e_2}| < \epsilon$ generates a classification of the set of subtrees. Each equivalence class is regarded as a state of the automaton, and each edge between the origins of two subtrees represents an edge between two states of the automaton. The transition probabilities between different states in this automaton are determined such that the original sequence is reconstructed in an ϵ-similar manner. Increasing l_1 and l_2 results in an increasing resolution of the dynamics.

4. ϵ-complexity: The *ϵ-complexity* C_ϵ is defined as the Shannon information of the state probabilities of the automaton.

For the construction of an ϵ-machine for a given symbol sequence Crutchfield and Young suggested a procedure to determine an "optimum" value of ϵ. However, their procedure does not guarantee that the given sequence is optimally reproduced by the corresponding automaton. For this purpose, the following technique estimating suitable tree parameters l_1, l_2, ϵ is proposed.

- For a given binary string the probabilities p_i of states $A_{i,n}^W$ of the n-cylinder partition P_n^{G*} are calculated.

- The construction of each ϵ-machine depends on a certain combination of tree length l_1, subtree length l_2, and ϵ. The state probabilities $\hat{p}_i$ of the n-cylinder partition $\hat{P}_n^{G*}(l_1, l_2, \epsilon)$ for the symbol sequences obtained from the corresponding ϵ-machine are compared with the state probabilities p_i of the original partition P_n^{G*} by calculating the Euclidian distance:

$$\Delta(\hat{P}_n^{G*}; P_n^{G*}) = [\sum_i (p_i - \hat{p}_i)^2]^{\frac{1}{2}} \tag{26}$$

- The complexity of the ϵ-machine with minimal Δ is defined as the ϵ-complexity of the given symbol string.

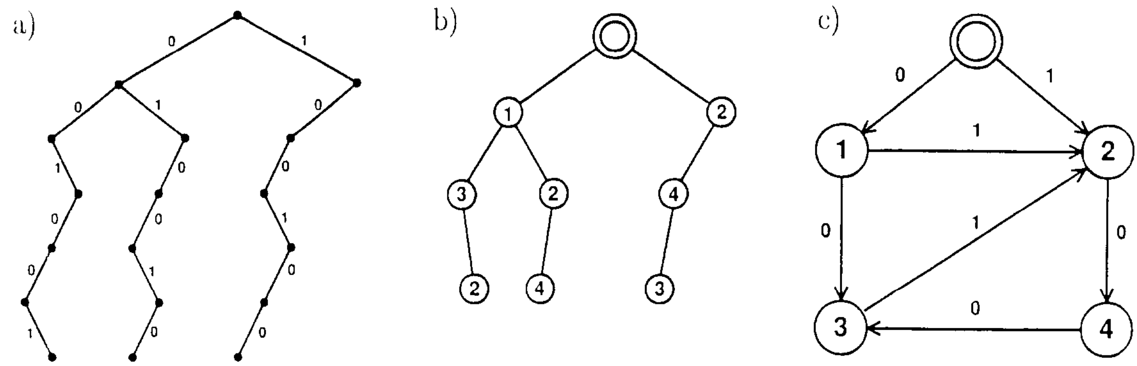


Fig. 2. An intelligible example to illustrate the procedure of calculating ϵ-complexity for the periodic sequence $S = 001001001001\ldots$. It is assigned to the binary tree of length $l_1 = 6$ (a). By definition of the equivalence classes (subtrees of length $l_2 = 3$) (b) the corresponding automaton (c) is derived.

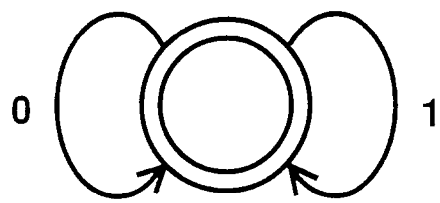


Fig. 3. Automaton for doubly stochastic behavior.

ϵ-complexity belongs to the class of dynamical measures, but the reason is not as easy to verify as for fluctuation complexity. For ϵ-complexity it is essential that it is defined on the level of a model, not on the level of the data themselves. The representation of the data (symbol sequence) by a model (the ϵ-machine) requires a mapping of states of the symbol sequence onto automaton states, which is not injective in general (for $\epsilon > 0$). Although ϵ-complexity is simply based on state probabilities on the level of the automaton, it refers to both transition probabilities and state probabilities on the level of the symbol sequence. Since the mapping between both levels is not injective, ϵ-complexity is not reducible to state probabilities on the level of the symbol sequence, hence it is characterized as DP^{G*}. (On the model level, ϵ-complexity would simply be a structural measure, since it is identical with $I^{(1)}$). More technical details on C_ϵ are given in [42].

- In case of periodic behavior with period $\wp$: $C_\epsilon = \log \wp$, if $\wp \leq l_2$.

  An intelligible example illustrating how the concepts of binary trees and equivalence classes are used to determine ϵ-complexity is given for the periodic sequence $S = 00100100100100\ldots$. This sequence can be assigned to a binary tree according to Figure 2a. By definition of the equivalence classes numbered by 0, 1, 2, 3, 4 (Figure 2b) an automaton as shown in Figure 2c is obtained consists of a closed loop composed of 3 automaton states, thus providing a complexity C

- In case of doubly stochastic behavior ($p_0 = p_1 = 0.5$, binary tree) the corresponding automaton only one state (Figure 3). The resulting similar sequence of automaton states has period such that its complexity vanishes: $C = 0$.

Table 1 Compact summary of section 2. Note that one-dimensional maps on an interval are considered. An asterisk in the last column indicates that the given values are only relevant in case of doubly stochastic behavior

section	complexity	classification	periodicity φ	stochasticity	
2.2.1	C_t	SP^{G*}	0	[illegible]	(*)
2.2.2	$I^{(q)}$	SP^{H}, SP^{G}, SP^{G*}	$\log \varphi$	$\log N$	
	$D^{(q)}$	SP^{H}	0		
	$K^{(q)}$	SP^{G}, SP^{G*}	0	$\lim_{n\to\infty} \frac{\log N_n}{n} = \log N_1$	(*)
2.2.3	$h_n^{(q)}$	SP^{G}, SP^{G*}	0	$\log \frac{N_{n+1}}{N_n} = \log N'$	(*)
	$\langle G \rangle$	SP^{H}, SP^{G}, SP^{G*}	0	$\log N'$	(*)
	$\langle M \rangle$	SP^{H}, SP^{G}, SP^{G*}	$\log \varphi$	$\log \frac{N}{\varphi}$	(*)
	EMC	SP^{H}, SP^{G}, SP^{G*}	$\log \varphi$	0	(*)
2.3	σ_Γ^2	DP^{H}, DP^{G}, DP^{G*}	0	0	
	C	DP^{G*}	$\log \varphi$	0	(*)

3. STRUCTURAL AND DYNAMICAL COMPLEXITY FOR THE LOGISTIC MAP

In this section a variety of different features of structural and dynamical kind, generated by the logistic map, is characterized by different complexity measures as they have been discussed in section 2. The basic intention of the present section is to show that and how different properties of different complexity measures are of different value in detecting different features. In a more fundamental sense this intention reflects an attempt to justify the non-universal and non-unique variety of existing complexity measures by the non-universal and non-unique variety of purposes for which they, respectively, may prove useful, suitable, or even well-adapted.

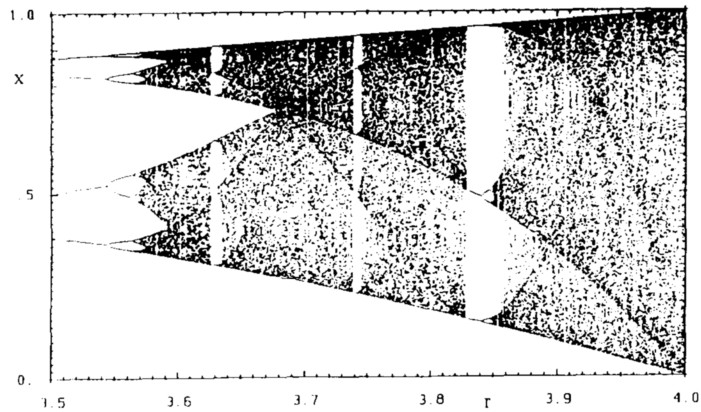


Fig. 4. bifurcation diagram for the logistic map in the parameter regime $r \in [3.5, 4]$.

3.1 The Logistic Map

One-dimensional discrete dynamical systems can show chaotic dynamics if they are noninvertible. A well-known example is the logistic map:

$$[0,1] \to [0,1], \quad x \mapsto F_r(x) = rx(1-x) \tag{27}$$

For $r \in [0,4]$ Eq. (27) is a map on the unit interval $I := [0,1]$ [43]. For $r > 4$ the iteration of Eq. (27) diverges for almost all initial values, which induces chaos on a Cantor set [44]. In this paper we consider the parameter regime $r \in [3.5, 4]$.

The *critical point* of the logistic map is $x_c = 0.5$. It is associated with a maximum value of $F_r(0.5) = \frac{r}{4}$. Thus Eq. (27) defines a surjective map only for $r = 4$. In case $r < 4$, the map is injective on the intervals $[0, 0.5], [0.5, 1]$, respectively. F_r is a *unimodal map*, which is monotonically increasing for $x < 0.5$ and monotonically decreasing for $x > 0.5$. Because the Schwarzian derivative is negative on I, F_r has only one attracting periodic orbit at most.

If one state of an orbit coincides with the critical point ($x_c = 0.5$), then the derivative is vanishing, $\frac{d}{dx}F_r^i(0.5) = 0$, and the orbit is called *superstable*. The functional dependence of superstable orbits on the control parameter r is represented by *supertrack functions* $s_i(r)$ [45]. In case of the logistic map, supertrack functions are continuous polynomials in r. They are recursively defined by $s_0(r) = 0.5$, $s_1(r) = F_r(0.5) = \frac{r}{4}$, $s_i(r) = F_r(s_{i-1}) = rs_{i-1}(1 - s_{i-1})$ for all $i = 1,2,3,...$ The bifurcation diagram in Figure 4 shows these functions $s_i(r)$ as dark lines. They are caused by the fact that iterates of x_c correspond to singularities in the natural measure of F_r.

As a function of r, the orbits of the logistic map provide different kinds of structural and dynamic behavior as it is reflected by the bifurcation diagram (Figure 4) as well as by the Ljapunov exponent. Below the so-called ***accumulation point*** $r_\infty = 3.569\ldots$ the Ljapunov exponent Λ does not exceed zero. This indicates period doubling behavior for $r < r_\infty$. Starting from a stable period-4 cycle at $r = 3.5$ [illegible] period-doubling bifurcation at $r_8 = 3.544\ldots$ leads to a stable period-8 orbit etc. This bifurcation scenario ends at the accumulation point r_∞ where the dynamics is periodic with period $\wp = 2^\infty$ and the associated attractor is given by a Cantor set.

Beyond r_∞ the Ljapunov exponent Λ increases to positive values characterizing chaotic regimes. *Periodic windows* with basic period k arise intermittently and separate these regimes, whenever the k-th iterate of F_r has k stable fixed points. The onset of periodic windows ($r = r_{k,\omega}$) with basic period k is defined by $s_1(r_{k,\omega}) = s_{k+1}(r_{k,\omega})$. Successive bifurcations generating harmonics with period $\wp = k \cdot 2^m (m = 0, 1, \ldots, k = 3, 4, \ldots)$ are found within each window. They are self-similar to the bifurcation scheme observed in the range $r \leq r_\infty$. Period-doubling cascades of a k-periodic window end at the accumulation point $r_{k,\infty}$ with $r_\infty := r_{1,\infty}$.

At certain values $r_{k,c}$ the dynamical behavior of the map changes discontinuously and qualitatively in such a way that the attractor is suddenly reduced from k subintervals to only one single interval (see Figure 4). This transition from chaos to chaos, known as ***interior crisis*** [46], is caused by the "collision" of the attractor with the corresponding unstable k-periodic orbit and defines the upper bound of the window. The period-3 window ($k = 3$) has been used to investigate the transition between chaos and order in detail. Here the third iterate of F_r has 3 stable fixed points. Periodic behavior with period $\wp = k = 3$ starts at $r_{3,\omega} = 1 + \sqrt{8}$, where the supertrack functions s_4 and s_1 intersect. The corresponding period-doubling behavior terminates at the accumulation point $r_{3,\infty} = 3.849\ldots$. As shown in Figure 4 an attractor consisting of $k = 3$ subintervals is created. The interior crisis at $r_{3,c} = 3.857\ldots$ [25] defines the upper bound of this period-3 window.

The qualitative behavior of the logistic map as a function of r within the period-3 window is self-similar to all other windows with $r \in [r_\infty, 4]$ and with basic periods $k = 3, 4, 5, \ldots$

In the range $r > r_\infty$, a so-called *reverse bifurcation sequence* describes the changing structure of the attractor of the logistic map. Any attractor in this range consists of 2^u subintervals (bands), and an aperiodic orbit meets these 2^u subintervals successively in a "bandperiodic" way. (A bandperiodic orbit with period 2^u falls into the same subinterval after 2^u time steps .) At *band merging points* $r_{u,*}$, $u = 1, 2, \ldots, \infty$, any existing 2^u bands join pairwise into 2^{u-1} bands. Band merging points are defined by the intersection of supertracks: $s_{3 \cdot 2^{u-1}}(r_{u,*}) = s_{4 \cdot 2^{u-1}}(r_{u,*})$. For example there is band merging from 4 to 2 bands at

$r_{2,*} = 3.592\ldots$, band merging from 2 to 1 occurs at $r_{1,*} = 3.678\ldots$. (For convenience, $r_{1,*} := r_*$ in the following.) The decreasing sequence $\{r_{u,*}\}_{u=1}^{\infty}$ converges geometrically toward the accumulation point r_∞, analogous to the increasing period doubling sequence.

For $r = 4$ the logistic map is surjective. The complete interval I is covered by the iteration points of F_4. The corresponding behavior is called fully developed chaos or exterior crisis.

3.2 Partitions

This section describes how the partitions P^H, P^G, P^{G*} and the corresponding state probabilities as well as transition probabilities are numerically generated in case of the logistic map.

3.2.1 Homogeneous Partition: P^H

For an investigation of structural complexity measures of the logistic map, the unit interval I is divided into $N^H = 1024$ bins of equal length $\varepsilon = 1/N^H$ providing the *homogeneous partition* $P^H = \{A_i^H\}_{i=1}^{1024}$. Using topological conjugacy of the tent map, the *natural measure* for the logistic map at $r = 4$ can be obtained analytically as $\mu(A_i^H) = \int_{A_i^H} \rho(x)dx$, where the probability density is given by $\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$. For arbitrary $r \in [3.5, 4]$, μ can be determined by numerical evaluation of the map for N_{it} iterations (after transients have died out) and plotting the result as a histogram. The normalized histogram becomes basically independent from a further increase of N_{it} at a ratio of $\frac{N_{it}}{N^H} = 10^4$, indicating a fairly good approximation. For a step width $\Delta r = 0.001$ of the control parameter r, $N^H = 1024$ is chosen as a suitable compromise between required CPU-time and numerical accuracy due to coarse–graining. Under these conditions the partition $P^H = \{A_i^H\}_{i=1}^{1024}$ allows to resolve periodicities up to $\wp = 16$.
By construction, P^H is homogeneous in state space but inhomogenous with respect to time. Therefore it does in general not represent a Markovian partition in the sense of the discussion in section 2.1. Nevertheless, abstract *transition probabilities* between states A_i^H, A_j^H can be defined from the geometrical properties of the map. It is useful to keep in mind that these transition probabilities do not reflect the concrete intrinsic dynamics of the system.

3.2.2 Generating Partition: P^G and P^{G*}

Using generating partitions, structural and dynamical complexity measures of type SP^G, DP^G, respectively SP^{G*}, DP^{G*}, can be investigated by discretizing the attracting state space in a manner designed by the dynamics of the system. Therefore, generating partitions are different for different systems. In this sense they account explicitly for the precise context of the dynamical system considered: they are contextual.

The *partition* P^G, which will be used in the following, is generated with respect to the decision point $d = 0.5$, separating the unit interval into two subintervals of equal width. There are two motivations for choosing d in this way: 1) $d = 0.5$ is the critical point of the map and therefore separates two subsets with different topological properties. 2) For the example of $r = 3.7$, Crutchfield and Packard [38] have demonstrated that the information production rate as a function of d is maximal for $d = 0.5$, thus determining the metric entropy $K^{(1)}$.

The partition P_1^G of first order ($n = 1$) is defined by the set $D_1 = \{d\}$. A refinement of this 2-cell partition P_1^G is constructed by successive calculation of the preimages $F_r^{-1}(d)$ of d. In this way, a refinement of order $n = 2$, denoted as P_2^G, is generated by the set $D_2 = \{F_{r,<}^{-1}(d), d, F_{r,>}^{-1}(d)\}$, where the second indices $<, >$ correspond to the preimage $x < x_c$ or $x > x_c$, respectively. In general, a n-th order refinement P_n^G is given by $D_n = \bigcup_{i=1}^{n-1} D_i \cup \{F_{r,<}^{-1}(d), F_{r,>}^{-1}(d) | d \in D_{n-1} \setminus D_{n-2}\}$.

Since the logistic map is a surjective 2:1 map only for $r = 4$, a n-th order refinement of the unit interval into $N_n^G = 2^n$ cells provides an upper limit of N^G for $r < 4$. It is easily recognized that a first order partition P_1^G, dynamically generated, is identical with a $N^H = 2$ cell partition of type P^H.

The *natural measure* on the partition P^G : $\mu(A_i^G)$, $i = 1, \ldots, N^G$, can numerically be derived from the measure on the partition P^H, such that a cell A_i^G is represented by an appropriately defined union of cells A_i^H. Because of discretization effects precise results require $N^G \ll N^H$.

In contrast to P^H, a generating partition P^G is Markovian in the sense that boundaries between cells are kept invariant by the dynamics of the system. This implies that the dynamics is considered with respect to a homogeneous flow of internal time, which results in an externally inhomogeneous distribution of cells. Therefore ***transition probabilities*** based on P^G refer to cells of different size.

The transition matrix, which determines the transition probabilities from state A_i^G to state A_j^G, $p_{i\to j}$, is derived analytically, as follows.

- Since any cell $A_{i_0}^G = [a, b]$ is dynamically mapped onto the subinterval $[F_r(a), F_r(b)]$ created by the Markovian generating partition, it can be considered as a union of two adjoint cells of P^G, say $A_{j_0}^G, A_{j_0+1}^G$. Then the transition probability is given by $p_{i_0\to j_0} = \frac{\text{length } A_{j_0}^G}{F_r(b) - F_r(a)}$, and $p_{i_0\to j_0+1} = 1 - p_{i_0\to j_0}$.

- As illustrated in the bifurcation diagram (Figure 4), the attractor of F_r is bounded by the supertrack functions $s_1(r)$ and $s_2(r)$. Consequently, the attractor is a subset of the unit interval I, whenever

$r < 4$. The transition matrix is therefore calculated in a way that excludes empty (with respect to the natural measure) cells outside the attractor.

- In periodic situations and in the interval $r_\infty < r < r_*$ there are cells A_j^G with vanishing measure on the interval $[F_r(\frac{r}{4}), \frac{r}{4}]$. In those cases where $\mu(A_{i_0}^G) \neq 0$ and for instance $\mu(A_{j_0}^G) = 0$, the "true" transition probabilities $p_{i_0 \to j_0} = 0$ and $p_{i_0 \to j_0+1} = 1$ are taken into account.

In the following, complexity measures based on P^G are calculated for a constant order $n = 6$ of refinement, producing a symbolic description of the logistic map. The cardinality N^G of the resulting partition varies as a function of r, since $\mu(A_i^G) > 0 \quad \forall i = 1, \ldots, N^G$. (This is, for instance, of influence for the determination of the topological entropy $K^{(0)} = \lim_{n\to\infty} \frac{\log N^G}{n}$.) For the purpose of the following applications, N^G will subsequently be considered as the number of cells A_i^G with nonvanishing measure. Then the number of cells N_n^G is equal to the number of cells N_n^W for given n and for partitions P_n^G, P_n^{G*}. Basically, all complexity measures considered (with the exception of fluctuation complexity σ_Γ^2, which will in detail be discussed later) are not sensitive to small variations of n.

B) The *n-cylinder induced partition* P^{G*} is required for measures of the complexity of explicit symbol sequences, e.g. for sequences of words of a language. Examples are algorithmic complexity, effective measure complexity, and ϵ-complexity. The basic partition P_1^G on the logistic map defines a binary alphabet such that the symbol 0 is assigned, if the iterated value of Eq. (27) falls into the interval $[0, 0.5]$. The symbol 1 is assigned if it falls into the interval $]0.5, 1]$. A n-cylinder induced partition P_n^{G*} of order n is then obtained by all words (substrings) $A_{i,n}^W (i = 1, \ldots, N_n^G)$ of length n that are generated by the map. P_n^{G*} and P_n^G are equivalent insofar as their natural measure and their transition probabilities are the same. They are different insofar as P_n^{G*} acts in the space Σ_F of symbol sequences, whereas P_n^G acts in the space of states of the map.

In order to describe the *natural measure* $p_i = \mu(A_{i,n}^W)$ by the relative frequencies of words $A_{i,n}^W$ in the finite symbol sequence S in a reliable manner, very long sequences are needed. The necessity for very long sequences is mainly due to the fact that the number of possible words increases exponentially with n in general. Of course, the structure of the symbolic sequence also influences the necessary sequence length L.

- *Algorithmic complexity*: Sequences of $L = 10^5$ turned out to be sufficient.
- *EMC*: Sequences of length $L = 10^6$ have been choosen to guarantee reliable results.
- *ϵ-machine*: Sequences up to lengths of 10^8 have been used.

3.3 Measures of Complexity for the Logistic Map

This section presents how different complexity measures, as introduced in section 2, reflect specific types of dynamical and structural features of the logistic map. The main features concerned are periodicity ($r < r_\infty$), onset of chaos (r_∞), band merging (r_*), period-3 window ($r_{3,a} < r < r_{3,e}$), and fully developed chaos ($r = 4$). Some of these features are investigated in those regimes where the necessary resolution in r is a minimum. Due to the self-similarity of the bifurcation diagram of the logistic map, they can in principle be found at infinitely many locations in parameter space.

During the following discussion it will be necessary to refer repeatedly to a set of figures that illustrates the behavior of the various complexity measures investigated. It is therefore most reasonable (and comfortable for the reader), to present these figures as a complete set, not distributed over the entire section. All diagrams show complexity as quantified by the respective measure as a function of the control parameter r of the logistic map. In detail, Figure 5 indicates algorithmic complexity C_a as an example of a non probabilistic structural measure. Figures 6a and 6b represent the Renyi dimensions $D^{(-10)}$ and $D^{(1)}$ as structural measures of type SP^H. Figures 7, 8, 9, and 10 show the metric entropy $K^{(1)}$, the mean information gain $< G >$, the mutual information $< M >$, and the effective measure complexity EMC. They are structural measures of type SP^G, SP^{G*}. The fluctuation complexity σ_Γ^2 shown in Figure 11 is a dynamical measure DP^G, DP^{G*}. The results for ϵ-complexity are given in Figure 12.

3.3.1 Periodicity

We start with the behavior of different complexity measures in case of periodic behavior. Structural measures as $I^{(q)}, D^{(q)}, K^{(q)}$ are capable of indicating periodic behavior, if the periodicity is resolved with respect to the state probabilities $p_i = \mu(A_i)$. For structural measures as $< G >, < M >, EMC$ (if they are formalized as depending on $p_{i \to j}$) and for dynamical measures of type DP^G, DP^{G*}, a corresponding resolution with respect to the transition probabilities is required in addition. This means that the underlying partition must be fine enough to describe the "true" system behavior in an ideally assumed continuous state space.

- *Algorithmic complexity* C_a vanishes for periodic behavior as it is shown in Figure 5.

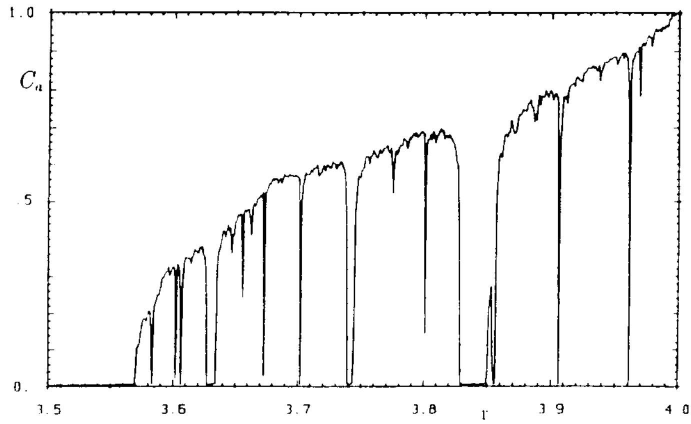
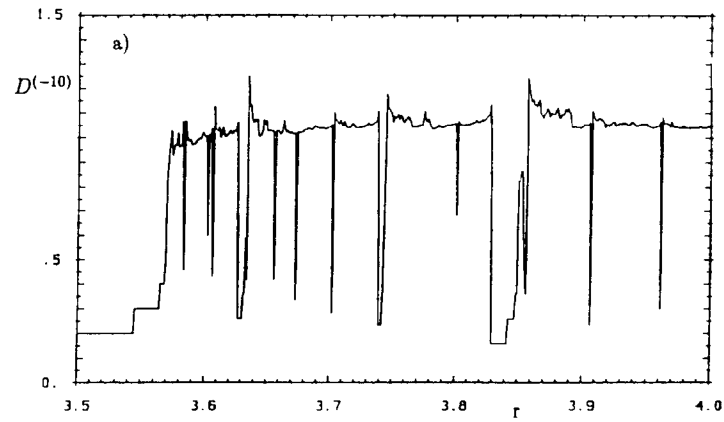


Fig. 5. Algorithmic complexity for sequences of length $L = 10^5$. The general shape of this curve is qualitatively reproduced already for $L = 10^4$.



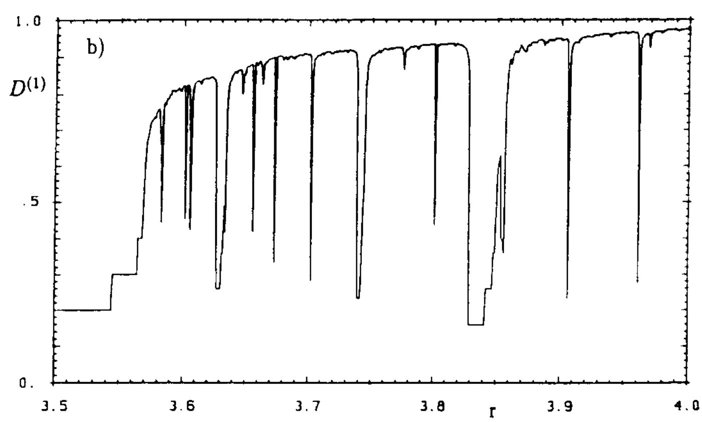


Fig. 6. Renyi-dimensions $D^{(q)}$ versus the control parameter r for $q = -10$ (a) and $q = 1$ (b). A partition $P^H = \{A_i^H\}_{i=1}^{1024}$ is used, and cells with $\mu(A_i^H) < 10^{-4}$ are not considered.

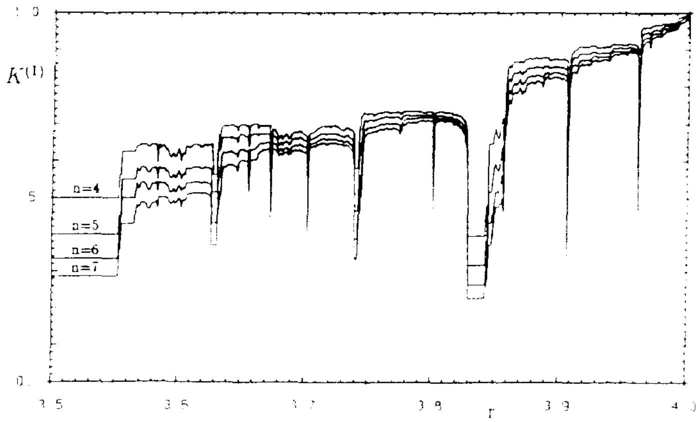


Fig. 7. Metric entropy for the logistic map calculated on P_n^G for $4 \leq n \leq 7$. The ratio $\frac{I_n^{(1)}}{n}$ converges to the metric entropy $K^{(1)}$ as a function of increasing n.

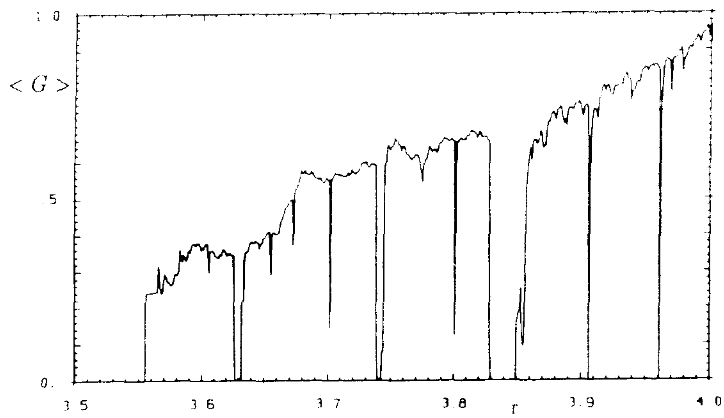


Fig. 8. Information gain $< G >$ calculated for P_6^G.

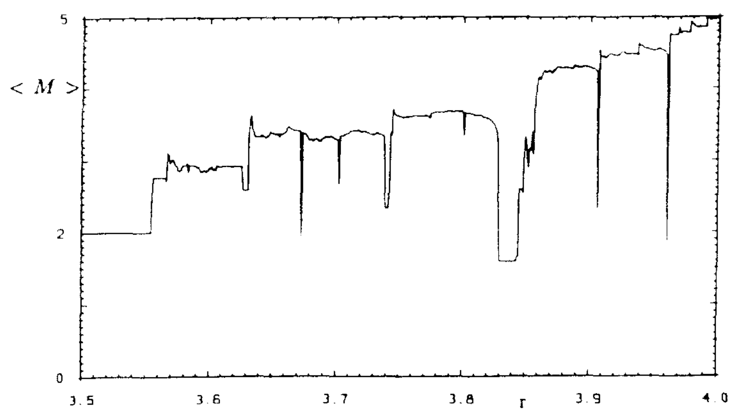


Fig. 9. Mutual information $< M >$ calculated for P_6^G.

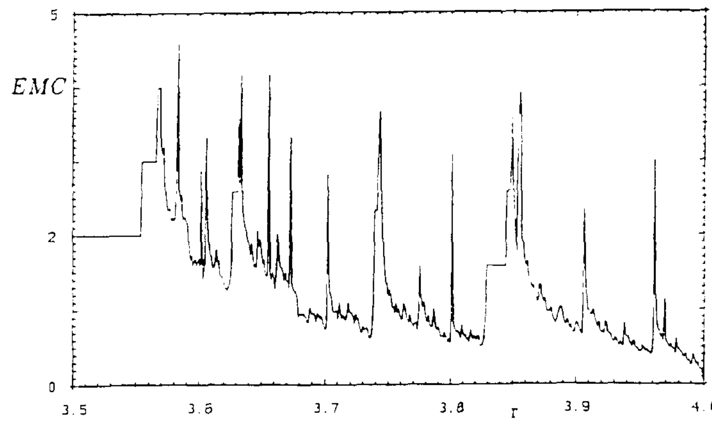


Fig. 10. Effective measure complexity calculated for P_{16}^{G*}. EMC vanishes for $r = 4$ and is given by $\log p$ for periodic behavior.

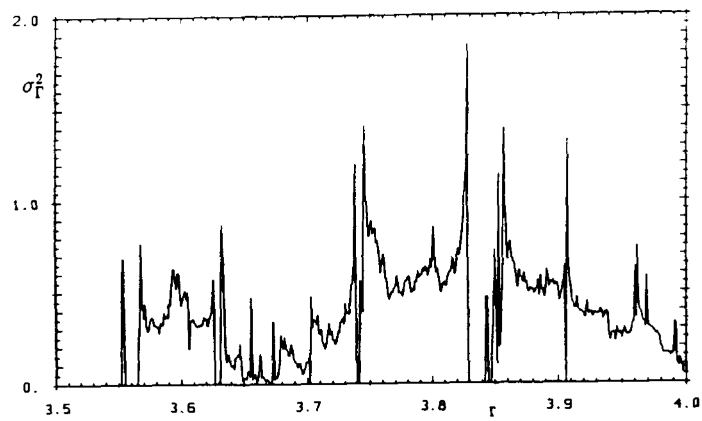


Fig. 11. Fluctuation complexity σ_Γ^2 for the logistic map as a function of r, calculated for P_6^G. It vanishes for regular behavior and for $r = 4$. The peaks at $r < 3.56, r \approx 3.8, r \approx 3.95$ correspond to unresolved periodic behavior in the distribution of state probabilities.

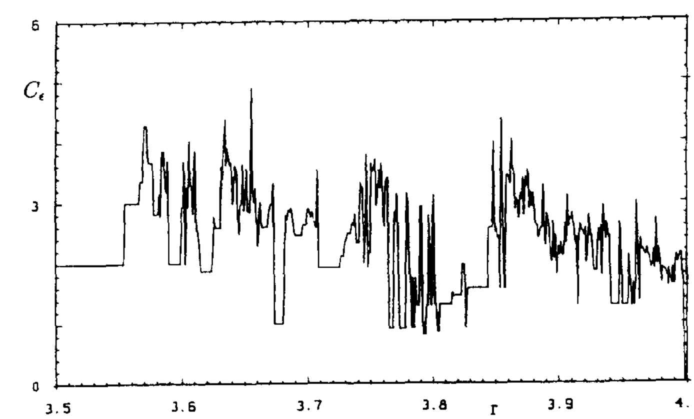


Fig. 12. ϵ-complexity calculated for P_8^{G*} and for the parameter combination $l_2 = 1, 2, \cdots, 8$; $l_1 = 2l_2$; $\epsilon = 0.02, 0.04, \cdots, 0.2$. The small complexity values in the parameter range [3.80, 3.83] are caused by the relative small lengths (l_1, l_2) of trees and subtrees.

- *Renyi-dimensions*: Figures 6a and 6b present the Renyi-dimensions $D^{(q)}$ for $q = 1$ (the so-called information dimension) and for $q = -10$. For an infinitely fine partition ($\varepsilon \to 0$), $D^{(q)}$ vanishes in any periodic case. For the finite partition P^H ($\varepsilon = 1/1024$), $D^{(q)}$ indeed tends to zero for [illegible] as well as within the periodic windows of the map. Moreover, a finite resolution is necessary in order to distinguish periodicities of different period $\wp$, since $D^{(q)} = \frac{\log \wp}{\log N^H}$ for any given [illegible]. Therefore, period doubling can be detected by discrete steps in Figures 6a,b.
  For $q < 0$, $D^{(q)}$ quantifies the scaling behavior of an "anticorrelation" function. If q [illegible], $D^{(q)}$ is determined by $min\{\mu(A_i^H)|\mu(A_i^H) > 0, i = 1, \ldots, N^H\}$ in the natural measure. Hence, [illegible] A_i^H with small natural measure $\mu(A_i^H) \neq 0$ provide the main contribution for moderately negative values of q. The plots shown in Figures 6a and b indicate an essentially identical behavior.

- *Metric entropy*: Since $K^{(1)} = \frac{\log \wp}{n}$ is only an approximation for finite $n (n = 6)$, small values of $K^{(1)}$ correspond to small periods $\wp$. For periodic behavior with prime period $\wp$, the expected value $K^{(1)} = 0$ is obtained in the limit $n \to \infty$. Comparing the behavior of $K^{(1)}$ as a function of [illegible] different periodic windows (Figure 7), a clear dependence of the speed of convergence on the period $\wp$ is observed.

- *Mean information gain*: In contrast to the complexity measures adressed so far, the information gain $< G >$ depends on transition probabilities $p_{i \to j}$. If the periodic behavior is resolved by the transition probabilities, it is characterized by $< G >= 0$, independent of period $\wp$ (see Figure 8).

- *Mutual information*: In order to be sensitive to periodicity, the corresponding period must be resolved by both, p_i and $p_{i \to j}$. If this requirement is fulfilled, the mutual information $< M >= \log \wp$ as shown in Figure 9.

- *Effective measure complexity*: As discussed in section 2.2.3, $EMC = \log \wp$ (Figure 10). Fairly large complexity is obtained for periodic behavior of high period, such that a clearcut distinction from chaotic behavior becomes difficult.

- *Fluctuation complexity*: If periodicity is resolved in the partition, then $\sigma_\Gamma^2 = 0$ independent of period $\wp$ (see Figure 11). If periodicity is not resolved in the partition P_n^G, then σ_Γ^2 shows a peak. This is understandable, since the corresponding "artificial" non-uniformity of the distribution of state probabilities produces an "artificially" small number of cells with non-vanishing measure $p_i = \mu(A_i^G)$ and $p_i \neq \frac{1}{\wp}$. As a consequence, the term $\log \frac{p_i}{p_j}$ in Eq.(24) increases and leads to a large value of fluctuation complexity.

- ϵ*-complexity*: Periodic behavior is indicated by non-vanishing values of complexity $C_\epsilon = \log \wp$, if the considered length of the subtrees $l_2 \geq p$ (Figures 2 and 12).

Although different complexity measures are sensitive to periodicities in a different way, periodic behavior is basically detected by all of them.

3.3.2 Accumulation Point: Onset of Chaos

Due to the finite discretization of the used partition ($n < \infty$), none of the complexity measures considered is capable of fixing the accumulation point ($p = \infty$) at $r_\infty = 3.569\ldots$ exactly. This corresponds to the fact that a finite partition does not permit the sensitivity of complexity measures to periodicities of arbitrarily high order. Therefore, the precise value of the complexity measure in question depends strongly on the refinement n, respectively ε, of the partition. This is consistent with the observation that the memory ν of a corresponding Markov process is very large at r_∞ (as well as $r_{k,\infty}$).

- C_a: As can be recognized in Figure 5, C_a vanishes for $r < r_\infty$, and the behavior for $r > r_\infty$ is characterized by a rapid and considerable increase of C_a.

- $D^{(q)}$: A similar increase applies to the generalized dimensions as shown in Figures 6a and 6b. The structure of the attractor of the logistic map at r_∞ is that of a Cantor set. Numerical estimates for its Hausdorff dimension, information dimension, and correlation dimension, repectively, provide $D^{(0)} = 0.538\ldots$, $D^{(1)} = 0.518\ldots$, and $D^{(2)} = 0.501\ldots$ [47, 48, 49]. These values are identical for all accumulation points $r_{k,\infty}$ of the logistic map.

- $K^{(q)}$: In Figure 7, r_∞ is indicated by a stop of the step-wise increase of $K^{(1)}$ as a function of r. A theoretical determination of the metric entropy $K^{(1)}$, based on non-chaotic orbits for $r = r_\infty$, yields $K^{(1)} = 0$ [1, 39].

- $< G >$: Mean information gain $< G >$ indicates r_∞ as the transition from $< G >= 0$ to a finite positive value (Figure 8).

- $< M >$: For periodic behavior mutual information is given by $< M >= \log \wp$. This leads to an infinite value of complexity for $\wp \to \infty$; it is indicated by a maximum value at $r = r_\infty$ in Figure 9.

- EMC: Accumulation points of period-doubling cascades are characterized by maximum values of EMC-complexity. For finite resolution ($n = 16$ in Figure 10) this maximum value is finite, whereas it becomes infinite in the limit $n \to \infty$ [1].

- σ_Γ^2: Fluctuation complexity σ_Γ^2 vanishes for periodic behavior of any resolved period. The [illegible] positive values for $r > r_\infty$ determines the accumulation point (Figure 11).

- C_ε: ε-complexity takes a maximum value at $r = r_\infty$ as shown in Figure 12. Due to theoretica[illegible] [illegible] diverges at accumulation points [16].

3.3.3 Band Merging

Among the considered complexity measures, only fluctuation complexity and ε-complexity are [illegible] to band merging.

- Figure 13a illustrates fluctuation complexity for partitions $P_n^{(i)}$ of different refinement n [illegible] Good convergence of σ_Γ^2 with increasing n can be observed for [illegible]. The graph of σ_Γ^2 [illegible] joins the corresponding graphs for $n = 5, 6, 7$ at $r \approx 3.8$, whereas the graph of [illegible] graphs of $n = 5, 7$ at $r \approx 3.75$. In the range of $r < 3.75$ it is clearly visible that [illegible] alternating behavior in the sense that σ_Γ^2 is large for odd n while it is small for even [illegible] This alternating behavior can be utilized as a criterion for band merging, here around r_2. [illegible] to understand this criterion, some detailed remarks are appropriate. (A discussion of the fu[illegible] of intricacies and subtleties is given elsewhere [50].)
  Consider the case $r < r_2$, where the attractor of the logistic map is split into two bands [illegible] range, the alternation of σ_Γ^2 with n is a consequence of the fact that the increasing refinement by increasing n applies alternatively to both bands. This follows by construction of the ge[illegible] partition, since the preimages of $x = 0.5$ that are created by successive values of [illegible] bands in an alternating sequence.
  Every even n provides a refinement corresponding to a distribution of state probabilities [illegible] *always* more homogeneous than a distribution of state probabilities for odd n. [illegible] any additional preimage of odd number introduces an additional cell only [illegible] The associated degree of non-homogeneity implies an additional contribution to σ_Γ^2. For this the discussion of σ_Γ^2 as a measure of complexity is based on even values of n.
  Nevertheless, the difference between even and odd values of n offers a sensitive criterion [illegible] merging, since the alternation gradually disappears as two bands of an attractor merge. [illegible] to illustrate this is obtained by plotting the difference of fluctuation complexity for two suc[illegible] n. Figure 13b shows such a plot for $\Delta\sigma := \sigma_\Gamma^2(n = 5) - \sigma_\Gamma^2(n = 6)$ as a function of r. The ma[illegible] of $\Delta\sigma$ turns out to correspond exactly to $r = r_2$. In the range $r < r_2$, $\Delta\sigma$ decreases until an

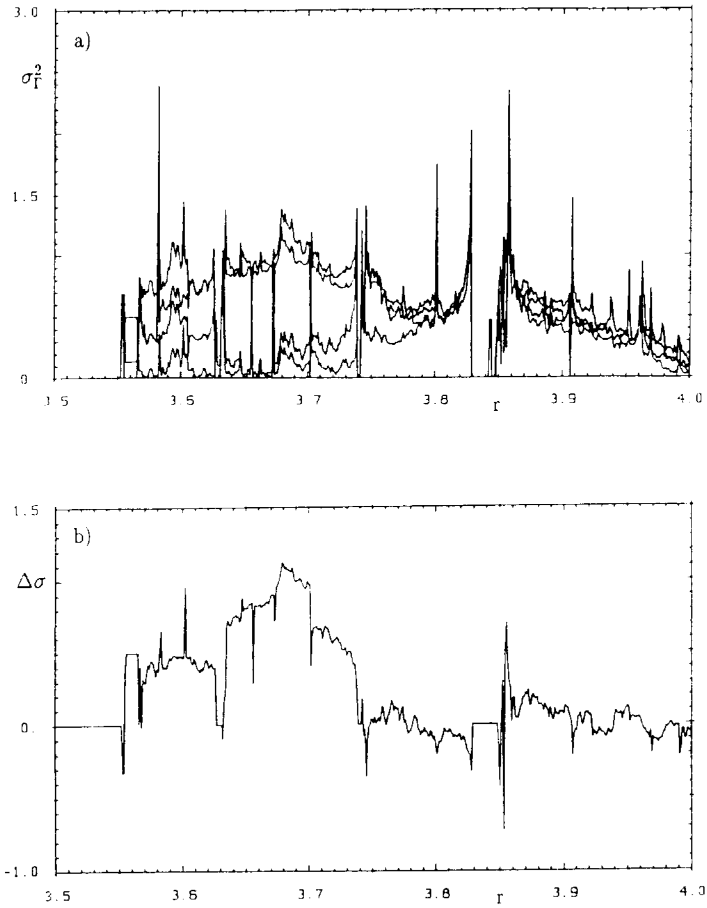


Fig. 13. a) Dependence of fluctuation complexity around band merging $n = 6, 4, 7, 5$ (from lowest to highest curve). b) $\Delta\sigma := \sigma_\Gamma^2(n = 5) - \sigma_\Gamma^2(n = 6)$ as a function of r provides a maximum at the band merging point r_*. Negative values of $\Delta\sigma$ arise due to unresolved features.

local maximum at $r = r_{2,*} = 3.592\ldots$ appears, where 4 bands merge into 2 bands, etc. For $r > r_*$, $\Delta\sigma$ declines with increasing r as both bands get effectively mixed such that inhomogeneities due to the refining procedure die out. Thus for a refinement $n \to \infty$ the alternation of σ_Γ^2 with n disappears immediately, as soon as r exceeds r_*. Many subtle effects are associated with the behavior of $\Delta\sigma$. They are discussed in detail in [50].

Summarizing, there are well-defined relationships between refinements of order n, the distribution of separate bands over the entire attractor, the alternating behavior of σ_Γ^2, and the sensitivity of σ_Γ^2 to band merging points as $r = r_*$. The same is true around $r_{k,*}$; for $k = 2$ this is indicated in Figure 13a.

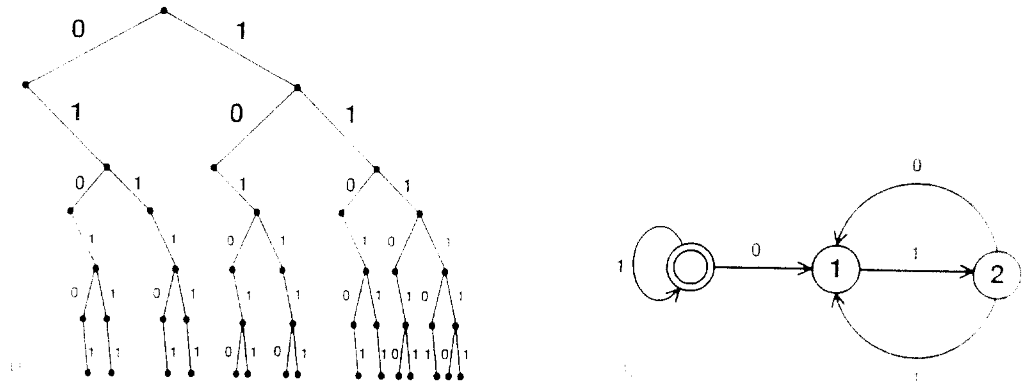


Fig. 14. Construction of the ϵ-machine at $r = r_*$: (a) binary tree of length $l_1 = 6$, and (b) corresponding automaton.

- *ϵ-complexity:*

  The sensitivity of ϵ-complexity for band merging has first been discussed in [16]. For the parameter $r_{k,*}$ the ϵ-complexity is given by $C_\epsilon = k$, if the condition $2^k \leq l_2$ holds. At $r = r_*$, the logistic map provides a sequence $S = 1s_1 1s_2 1s_3 1s_4 \ldots$, where s_i can take values 0 or 1 in a random way. This sequence indicates an alternation between band1 and band2, and it leads to a binary tree of length $l_1 = 6$ (see Figure 14a). By definition of the equivalence classes the corresponding automaton (Figure 14b) consists of only two states, both with state probability $p_i = 0.5, i = 1, 2$. This provides $C_\epsilon = 1$.

3.3.4 Period-3 Window

For the investigation of the logistic map in the period-3 window ($3.82 \leq r \leq 3.87$) an increased resolution of $\Delta r = 10^{-4}$ in parameter space is used. As particular features within the period-3 window, its onset, the period-3 accumulation point, the 3-band attractor, and the feature of interior crisis will be considered.

- *Onset of period-3 window:* At the onset of the period-3 window ($r_{3,o} = 1 + \sqrt{8}$) a significant peak in complexity is only found using fluctuation complexity. All other complexity measures $C_\mu, D^{(q)}, K^{(q)}, < G >, < M >, EMC, C_\epsilon$ are not sensitive to this type of transition. They simply approach their specific periodic limit.

  The peak of σ_Γ^2 in Figure 15 at $r_{3,o}$ corresponds to a highly non-uniform distribution of state probabilities (measure $\mu(A_i^G)$) at the transition from the chaotic state, which exhibits a countably infinite number of singularities, to the periodic state, which covers exactly 3 singularities. σ_Γ^2 is sensitive to this type of non-uniformity, caused by tangent bifurcation at intermittency.

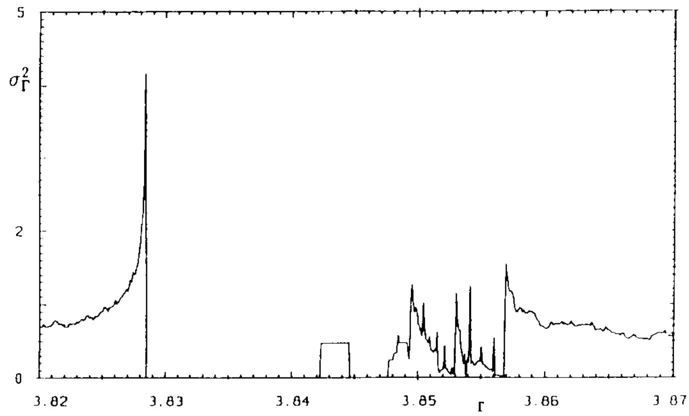


Fig. 15. Fluctuation complexity, calculated for P_6^G, in the parameter regime $r \in [3.82, 3.87]$ (period-3 window). The transitions from periodicity to chaos and vice versa are investigated with resolution $\Delta r = 10^{-4}$. Positive values of σ_Γ^2, e.g in the neighborhood of $r = 3.843$, correspond to unresolved periodicities in the state probabilities.

- *Accumulation point:*

  All considered complexity measures are more or less sensitive to the period-3 accumulation point $r_{3,\infty} = 3.849\ldots$. Slight differences in sensitivity are due to the same kind of problems as already mentioned for the accumulation point r_∞.

- *3-band attractor:*

  For $r > r_{3,\infty}$ some marginally resolved periodic "sub"windows within the period-3 window are characterized by low complexity.

- *Interior crisis:*

  The transition "from chaos to chaos" at $r_{3,c} = 3.856\ldots$ [25] corresponds to a discontinuous increase of N, the number of cells A_i, with nonvanishing measure. Therefore, all considered complexity measures, with the only exception of EMC, increase rapidly as soon as r reaches $r_{3,c}$.

  Fluctuation complexity increases sharply at $r = r_{3,c}$ and declines for $r > r_{3,c}$. The increase at $r = r_{3,c}$ is a consequence of the increase in N^G as well as of the extremely inhomogeneous distribution of state probabilities for $r \approx r_{3,c}$. With a further increase of r, fluctuation complexity declines since the measure on the attractor tends to get more uniform.

  Renyi-dimensions for $q < 0$ show a similar behavior for $r \geq r_{3,c}$, since they correspond to "anticorrelations" in the natural measure of P^H (Figure 6a).

3.3.5 Fully Developed Chaos

At $r = 4$ we have the limiting case of random behavior (with vanishing memory, $\nu = 0$) on the one interval. Hence, complexity à la Kolmogorov and Chaitin will take its maximum here, whereas complexity as introduced according to Figure 1 vanishes. A look at Figures 5-12 shows that all considered measures with the exception of σ_Γ^2, C_λ and EMC indicate increasing complexity as a function of r as a general tendency (which is interrupted by ranges of small complexity for regular behavior). Complexity takes its maximum value for $r = 4$. Thus complexity measures as $< G >$, $< M >$, $K^{(q)}$, $D^{(q)}$ basically express the degree of randomness as it is the case for algorithmic complexity.

- C_A: By definition algorithmic complexity is the prototype of a measure of randomness. Its maximum $C_A = 1$ is obtained for $r = 4$.

- $D^{(q)}$: Based on the analytical expressions for the natural measure μ at $r = 4$, Renyi-dimensions can be approximated as:

$$D^{(q)} = \begin{cases} 1 & \text{for} \quad q \le 2 \\ \frac{q}{2(q-1)} & \text{for} \quad q \geq 2 \end{cases}$$

  The discontinuity at $q = 2$ is a consequence of the non-hyperbolicity of the logistic map at $x = 0.5$ and of the fact that this critical point is mapped onto the two values 0 and 1 only, i.e. the measure has two singularities at $x = 0, 1$. This situation can be considered as an analog to a thermodynamical phase transition [51]. Thus $D^{(q)}$ is not a strictly decreasing function of q, and, in contrast to chaotic behavior in the range $r < 4$, the attractor is not a multifractal for $r = 4$ [25].

- $K^{(q)}$: On a generating partition P^G the natural measure is uniformly distributed for $r = 4$ and arbitrary n: $\mu(A^G_{i,n}) = 1/N^G_n = 2^{-n} \quad \forall \quad i = 1, \ldots, N^G_n$. Therefore, the generalized entropies are given by $K^{(q)} = 1$ for all q. Thus, the phase transition of the structural complexity measure $D^{(q)}$ (of type SP^H) has no counterpart in the structural complexity measure $K^{(q)}$, which is of type SP^G.

- $< G >$: Since the logistic map shows doubly stochastic behavior with $N' = 2$, $< G > = \log N' = 1$ for fully developed chaos.

- $< M >$: For doubly stochastic behavior mutual information is given by $< M > = \log \frac{N}{N'}$. Because F_4 is determined by $N' = 2$ and $N = 2^n$ we obtain $< M > = n - 1$ for this case. As a consequence mutual information vanishes for $n = 1$. In this case the symbolic dynamics of the logistic map is equivalent to a coin-tossing process, which is completely random as defined in section 2.1. Among all complexity measures considered here, only mutual information can identify a completely random process as a specific case of doubly stochastic behavior.

- EMC: Analogous to the partition P_n^G the states according to P_n^{G*} are uniformly distributed for $r = 4$. Consequently $EMC = 0$.

- σ_Γ^2: Fluctuation complexity vanishes for $r = 4$, since the measure $\mu(A_i^G)$ is uniformly distributed for all subintervals $A_i^G \in P^G$. This behavior is similar to the situation at $r = r_*$, since the dynamics of F_r at $r = 4$ is selfsimilar to the dynamics of F_r^2 at $r = r_*$.

- C_ϵ: The dynamics of F_4 generates a completely uncorrelated binary sequence. Thus the corresponding binary tree of length l_1 consists of all combinatorially possible words of length $n = l_1$, providing exactly one equivalence class and therefore vanishing ϵ-complexity (Figure 3 and 12).

Table 2: Sensitivity of the considered complexity measures for specific types of behavior in case of the logistic map. In this table theoretically or numerically approximated values of complexity are given with the corresponding references. For $n \to \infty$, C_a and $K^{(1)}$ coincide. An asterisk $(*)$ in the last column indicates that the given values are only relevant in case of doubly stochastic behavior.

complexity	periodicity	$r_\infty =: r_{1,\infty}$	$r_* =: r_{1,*}$	$r_{3,c}$	$r = 4$
C_a	0	0	0.5	increase	1 $(*)$
$I^{(q)}$	$\log \wp$	singular		increase	$\log N$
$D^{(q)}$	0	$q = 0 : 0.538$		$q > 0$:	$q \leq 2 : 1$
		$q = 1 : 0.518$		increase	$q \geq 2 : \frac{q}{2(q-1)}$
		$q = 2 : 0.501$		$q < 0$:	
		[48, 49]		peak	
$K^{(1)}$	0	0 [39]	0.5	increase	1 $(*)$
$< G >$	0			increase	1 $(*)$
$< M >$	$\log \wp$	singular		increase	$n - 1$ $(*)$
EMC	$\log \wp$	singular			0 $(*)$
σ_Γ^2	0		alternating	peak	0
C_ϵ	$\log \wp$	singular	1		0 $(*)$

4. SUMMARY

The central subject of this paper is an attempt to classify various existing complexity measures into a four-fold scheme based on the dichotomous notions of structure (S) and dynamics (D) as well as homogeneous partitions (P^H) and generating partitions (P^G). The four classes of measures resulting from this scheme are denoted as SP^H, SP^G, DP^H, and DP^G. Although there is no "best" definition of complexity in a unique manner, the presented classification scheme is intended to facilitate better orientation within the huge set of existing complexity measures.

Structural aspects of a point set are reflected by the appearance of state probabilities p_i (with respect to the partition) in the definition of the measure. Dynamical measures contain transition probabilities $p_{i\to j}$ in addition. In some cases it turns out that the formal definition of a dynamical measure can be rephrased such that it does no longer contain transition probabilities explicitly. This issue raises the question of irreducibility in the context of the suggested scheme. Within the given classification, measures are considered as dynamical measures if transition probabilities in their formal definition are not reducible to state probabilities. Otherwise, they are structural measures.

Only two measures out of the investigated set are dynamical measures of type DP^G in this sense: fluctuation complexity and ϵ-complexity. Structural measures with reducible transition probabilities (SP^G) are dynamical entropies, information gain, mutual information, and effective measure complexity. Algorithmic complexity, generalized informations, and generalized dimensions are structural measures, whose definition is totally independent of transition probabilities.

It is apparent that classificational ambiguities due to reducibility of dynamical elements appear solely for those measures defined on generating partitions. The reason is that a generating partition by construction contains the dynamics of a system implicitly. For this reason, it is in principle possible to "cover" particular dynamical aspects by such a partition, thus providing measures of type SP^G. Nevertheless, there remain cases of irreducible DP^G measures as mentioned above. A well-defined general criterion for irreducibility has not yet been found.

Application of the set of investigated complexity measures to the logistic map shows that particular measures are required to detect particular features of the map. The specific differences between complexity measures in this respect are summarized in Table 2. It is also important to note that there are substantial discrepancies between homogeneous and generating partitions. In case of the logistic map, this can most clearly be seen at fully developed chaos ($r = 4$). Refinement of a generating partition ($n \to \infty$) does in general provide measures that are different from those obtained from an identically refined homogeneous partition.

Complexity measures of type SP^H assign highest complexity to random behavior. In contrast, measures of type DP^G vanish for random behavior. For regular (stationary, periodic) behavior, measures of both types either vanish or are given by $\log \wp$. High complexity according to DP^G- (and some SP^G-) measures corresponds to specific kinds of more sophisticated dynamical behavior, e.g., the onset of chaos.
In this respect SP^H and DP^G can be considered as classes of complexity measures accounting for the basic two notions of complexity indicated in the introduction. For measures of type SP^G this clear distinction is lost. Some of those measures increase with randomness, others do not. Some of them vanish for regular behavior, others do not.
As a final remark, we should like to add a brief comment on the relationship between the concepts of complexity and meaning as it has been proposed by several authors [1, 52, 53]. In a recent publication [41] we have pointed out to some detail, how both concepts might be regarded as corresponding to each other conceptually and operationally. Within the classifying scheme presented here, this correspondence is restricted to complexity measures of type DP^G, in particular to fluctuation complexity σ_Γ^2.

Acknowledgements - We would like to express our thanks for stimulating and helpful discussions with Jim Crutchfield, Mats Nordahl, and Gregor Morfill. Support concerning computational work has been given by Valentin Demmel and Udo Schwarz.
Part of this paper has been completed during a research visit of one of us (H.A.) at the Santa Fe Institute. This visit was based on NATO grant # SA.9-15-04 (RV.910588). It is a pleasure to thank the staff and the members of the Santa Fe Institute for their hospitality.
This work has been supported by BMFT grant # 05 5ME94A,E.

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