Matrix-measure criterion for synchronization in coupled-map networks

Ping Li,[1,2] Maoyin Chen,[3] Ye Wu,[2] and Jürgen Kurths[4,5]

[1]*University of Electronic Science and Technology of China, Chengdu 610054, People's Republic of China*

[2]*Institute of Physics, University of Potsdam, Am Neuen Palais 10, D-14469 Potsdam, Germany*

[3]*Department of Automation, TNlist, Tsinghua University, Beijing 100084, People's Republic of China*

[4]*Potsdam Institute for Climate Impact Research (PIK), Telegraphenberg A 31, 14473 Potsdam, Germany*

[5]*Institute of Physics, Humboldt University, 10099 Berlin, Germany*

(Received 31 March 2009; published 8 June 2009)

We present conditions for the local and global synchronizations in coupled-map networks using the matrix measure approach. In contrast to many existing synchronization conditions, the proposed synchronization criteria do not depend on the solution of the synchronous state and give less limitation on the network connections. Numerical simulations of the coupled quadratic maps demonstrate the potentials of our main results.

DOI: 10.1103/PhysRevE.79.067102 PACS number(s): 89.75.−k, 82.40.Bj

Synchronization, as one of the most remarkable phenomena that arise in many fields of sciences, ranging from natural to social systems, has attracted a lot of attention in a few coupled subsystems but recently also in networks with complex topology [1–13]. It has been shown that the topological properties of coupled dynamical systems affect crucially the synchronizability of subsystems in a network [3,6,12]. These topological properties include the small-world [14] property and scale-free [15] nature of the connectivity distribution, typical for many real world networks. Random small-world and scale-free networks are generally better synchronizable than regular networks [3]. Moreover, systems with a homogenous connectivity are better synchronizable than heterogeneous ones. Until now, many criteria for network synchronization have been derived (e.g., [16–19]).

Pecora and Carroll [3] proposed the *master-stability-function* (MSF) method to study complete synchronization of coupled dynamical systems with complex network connections. Its main idea is to transform the stability of the synchronous manifold into the stability of the corresponding master-stability equation. It is valid for small perturbations around the synchronous state. Another choice to derive synchronization conditions is the Lyapunov direct method by which one can construct a Lyapunov function and then analyze the local or global synchronization [12,20–22]. Recently, the *connection-graph stability* was proposed in [13] to provide global synchronization conditions by combining graph properties.

The previous results obtained by these methods, to some extent, have their limitations. The MSF method requires that the Laplacian matrices are diagonal or block diagonal. Some of the existing local stability criteria (such as [22]) are based on the prerequirement of a synchronous state. Here we develop local and global synchronization conditions for much wider applications, especially without assuming that the coupling matrix is non-negative and diagonal. By means of the *matrix measure approach* [8,9], we achieve synchronization criteria independent of the uncoupled solution of the synchronous state.

We study the following coupled-map networks:

$$x_i(t+1) = f[x_i(t)] + \varepsilon \sum_{j=1}^{N} W_{ij}\{f[x_j(t)] - f[x_i(t)]\} \tag{1}$$

for $1 \le i \le N$, where $f[x_i(t)]$ is a continuously differential function governing the dynamics of the individual nodes, often chosen to be a chaotic map. $W=(W_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ describes the coupling configuration: if there is a connection between node i and node j, then $W_{ij} \neq 0$; otherwise $W_{ij}=0$. Here W is not restricted to be completely symmetric and non-negative. The network becomes synchronized if $\lim_{t\to\infty}|x_i(t)-x_j(t)|=0$ for all $1 \le i, j \le N, i \neq j$. In this case, each node evolves in the same manner, i.e., $s(t+1) = f[s(t)]$ and $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$. Denote $\mathbf{S}=\{x = (x_1,x_2,\ldots,x_N)^T, x_i \in R, x_i=x_j, i,j=1,2,\ldots,N\}$ be the synchronization manifold. Without loss of generality, let $x_1(t)$ be the reference synchronized direction. Then we define the stability of the synchronization manifold as follows: (i) system (1) is said to be *locally synchronized* if there exists a constant $\delta>0$ such that if $\|x_i(t_0)-x_1(t_0)\|<\delta$ for $1\le i\le N$, then for arbitrary $\epsilon>0$, there exists a constant $T>t_0$ such that $\|x_i(t) - x_1(t)\|<\epsilon$ for all $t\ge T$ and $1\le i\le N$ and (ii) system (1) is said to be *globally synchronized* if for arbitrary $\epsilon>0$, there exists a constant $T>t_0$ such that $\|x_i(t)-x_1(t)\|<\epsilon$ for all $t\ge T$ and $x_i(t_0)\in R, 1\le i\le N$.

Denote by $X_{1i}(t)=x_i(t)-x_1(t)$, then

$$X_{1i}(t+1) = f[x_i(t)] - f[x_1(t)] + \varepsilon \sum_{j=1}^{N} W_{ij}(f[x_j(t)] - f[x_i(t)]) - \varepsilon \sum_{j=1}^{N} W_{1j}(f[x_j(t)] - f[x_1(t)])$$

for $2 \le i \le N$. Define the matrix S_w,

$$S_w = \begin{bmatrix} -\left(W_{12} + \sum_{j\neq 2} W_{2j}\right) & W_{23} - W_{13} & \cdots & W_{2N} - W_{1N} \\ W_{32} - W_{12} & -\left(W_{13} + \sum_{j\neq 3} W_{3j}\right) & \cdots & W_{3N} - W_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N2} - W_{12} & W_{N3} - W_{13} & \cdots & -\left(W_{1N} + \sum_{j\neq N} W_{Nj}\right) \end{bmatrix}.$$

Then, we get one compact form,

$$\bar{X}(t+1) = (I_{N-1} + \varepsilon S_w)\bar{f}[x(t)], \tag{2}$$

where $\bar{X}(t)=[X_{12}(t),X_{13}(t),\ldots,X_{1N}(t)]^T$ and $\bar{f}[x(t)] =\{\bar{f}_1[x_2(t)],\ldots,\bar{f}_{N-1}[x_N(t)]\}=\{f[x_2(t)]-f[x_1(t)],\ldots,f[x_N(t)] -f[x_1(t)]\}^T$.

We first introduce the concept of matrix measure. Let C denote the field of complex numbers; the matrix measure of a complex square matrix $B\in C^{n\times n}$ is defined by [23]

$$\mu(B) = \lim_{h\to 0^+} \frac{\|I_n + hB\| - 1}{h}, \tag{3}$$

in which $\|\cdot\|$ is a matrix norm and I_n is the identity matrix. For the matrix norms $\|B\|_1=\max_j\Sigma_{i=1}^N|b_{ij}|$, $\|B\|_2=\sqrt{\lambda_{\max}(B^TB)}$, and $\|B\|_\infty=\max_i\Sigma_{j=1}^n|b_{ij}|$, we obtain the matrix measure [23]: $\mu_1(B)=\max_j\{\mathrm{Re}(b_{jj})+\Sigma_{i=1,i\neq j}^n|b_{ij}|\}$, $\mu_2(B)=\frac{1}{2}\lambda_{\max}(B^H+B)$, and $\mu_\infty(B)=\max_i\{\mathrm{Re}(b_{ii})+\Sigma_{j=1,j\neq i}^n|b_{ij}|\}$, respectively, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a complex matrix and B^H is the complex-conjugate transpose of a complex matrix. Note that $B^H=B^T$ if B is a real square matrix.

To obtain the condition for local synchronization, we consider small perturbations $\eta_i(t)(1\leq i\leq N-1)$ near the reference direction of the synchronization manifold. Then $\bar{f}_i[x_{i+1}(t)]$ can be approximated by

$$\bar{f}_i[x_{i+1}(t)] = f'[x_1(t)]\eta_i(t), \tag{4}$$

where $f'[x_1(t)]$ is the derivative at the reference synchronized direction. Then Eq. (2) is rewritten as

$$\eta(t+1) = f'[x_1(t)](I_{N-1} + \varepsilon S_w)\eta(t), \tag{5}$$

where $\eta(t)=[\eta_1(t),\eta_2(t),\ldots,\eta_{N-1}(t)]^T$. Next we construct a non-negative function,

$$V(t) = \eta^T(t)P\eta(t), \tag{6}$$

where P is an arbitrary positive definite matrix. Clearly, $V(t)\geq 0$ and the equality holds only if all components of $\eta(t)$ are equal to zero. That is, the synchronization errors with respect to the reference direction will disappear as $V(t)$ converges to zero. Note that P can be decomposed into $P=M^TM$, where M is an $(N-1)\times(N-1)$ nonsingular square matrix. Then

$$V(t) = \eta^T(t)M^TM\eta(t) = f'[x_1(t-1)]^2\eta^T(t-1)M^T \times[M^{-T}(I_{N-1} + \varepsilon S_w)^TM^TM(I_{N-1} + \varepsilon S_w)M^{-1}]M\eta(t-1). \tag{7}$$

We introduce $U=M^{-T}(I+\varepsilon S_w)^TM^TM(I+\varepsilon S_w)M^{-1}$. Then

$$\begin{aligned} V(t) &\leq f'[x_1(t-1)]^2\mu_\theta(U)V(t-1) \\ &= \prod_{k=0}^{t-1}\{f'[x_1(t)]^2\mu_\theta(U)\}V(0), \end{aligned} \tag{8}$$

where $\mu_\theta(U)$ is the matrix measure of U [23] and $\theta \in\{1,2,\infty\}$. Thus if

$$\lim_{t\to\infty}\prod_{k=0}^{t-1}|f'[x_1(k)]|\sqrt{\mu_\theta(U)} = 0, \tag{9}$$

then

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \eta^T(t)M^TM\eta(t) = 0.$$

Accordingly, the synchronization error $\eta(t)$ can be asymptotically stable. Hence Eq. (9) holds if

$$\lim_{t\to\infty}\frac{1}{t}\log\prod_{k=0}^{t-1}|f'[x_1(k)]|\sqrt{\mu_\theta(U)} = 0, \tag{10}$$

that is,

$$\log\sqrt{\mu_\theta(U)} + \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\log|f'[x_1(k)]| = 0. \tag{11}$$

Then a sufficient condition for local synchronization is

$$\mu_\theta(U) \leq e^{-2\lambda^f}, \tag{12}$$

where $\lambda^f=\lim_{t\to\infty}\frac{1}{t}\Sigma_{k=0}^{t-1}\log|f'[x_1(k)]|$ is the Lyapunov exponent of the map f. For a chaotic map, the Lyapunov exponent can be calculated. Therefore, the upper bound of the matrix measure can be determined in Eq. (12). Further, the proposed condition can be used to evaluate the stability of the synchronization manifold of coupled-map networks.

In Ref. [22], the authors presented a local stability condition for synchronization in which the norm of the Jacobian is related to the synchronization trajectory (a solution of the uncoupled system). However, it is well known that the synchronization trajectory is unknown in advance, and the criterion is hard to be used in applications. As for the condition given by Eq. (12), the upper bound of the matrix measure is only controlled by the Lyapunov exponent of the map f which can be calculated independently. In addition, the condition is less conservative than those in Refs. [17,18] where the coupling matrices are diagonal or block diagonal. Moreover, the coupling in this Brief Report can be cooperative or competitive, i.e., $W_{ij}>0$ or $W_{ij}<0$. Thus the condition derived from the matrix measure has a wider range of application.

The above analysis is based on the linear expansion around the synchronization manifold, which is valid only for small perturbations around the synchronization manifold. To derive a global criterion that guarantees synchronization of the coupled-map networks for arbitrary initial values, we assume a basic property for f,

$$|f(x)-f(y)| \leq \sup|f'||x-y|. \tag{13}$$

A natural way is to choose a non-negative function for Eq. (2) as

$$V(t)=\bar{X}^T(t)P\bar{X}(t), \tag{14}$$

and we can derive a condition under which

$$V(t)\to 0 \quad \text{as} \quad t\to\infty. \tag{15}$$

We get then

$$\begin{aligned} V(t+1) &= \bar{X}^T(t+1)P\bar{X}(t+1) \\ &= \bar{f}^T[x(t)](I_{N-1}+\varepsilon S_w)^T P(I_{N-1}+\varepsilon S_w)\bar{f}[x(t)] \\ &\leq \mu_\theta(U)\bar{f}^T[x(t)]M^T M\bar{f}[x(t)] \leq \mu_\theta(U)\sup|f'|^2 V(t). \end{aligned} \tag{16}$$

Then

$$V(t)\leq V(0)[\mu_\theta(U)\sup|f'|^2]^t. \tag{17}$$

Therefore, network (1) globally asymptotically synchronizes if

$$\mu_\theta(U)\leq \frac{1}{\sup|f'|^2}. \tag{18}$$

Above criterion (18) allows us to consider any solution $x(t)$ of Eq. (1).

To verify our criteria, we took a quadratic map for the unit dynamics, that is, $f(x)=ax(1-x), x\in[0\ \ 1]$. The Lyapunov exponent for this map can be directly calculated for constant $a\in[3\ \ 4]$. As the simplest case, the globally coupled network with $N=100$ is considered. The coupling matrix has then the form

$$W=\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix},$$

which yields

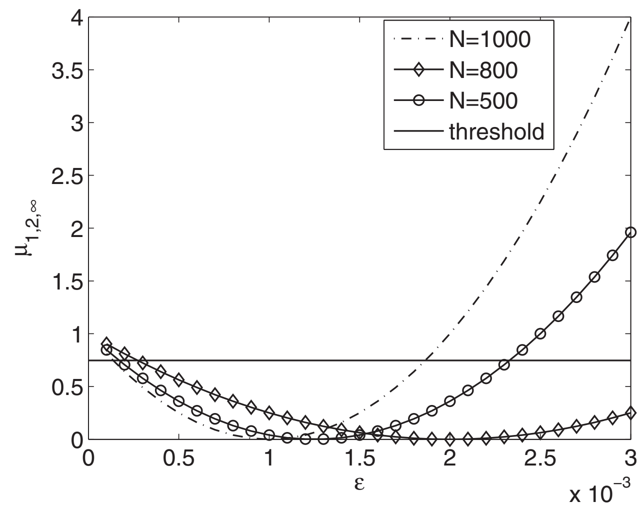

FIG. 1. The matrix measure vs coupling strength in the global coupled networks for different scales. In this case, $a=3.59$.

$$S=\begin{bmatrix} -N & 0 & \cdots & 0 \\ 0 & -N & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -N \end{bmatrix}.$$

We take M as the identity matrix (i.e., $M=I_N$) and get $U=I+\varepsilon(S_w^T+S_w)+\varepsilon^2 S_w^T S_w$. From the definition of the matrix measure [23], we have

$$\mu_{\{1,2,\infty\}}[I+\varepsilon(S_w^T+S_w)+\varepsilon^2 S_w^T S_w]=(\varepsilon N-1)^2. \tag{19}$$

So we choose an $\varepsilon\sim\frac{1}{N}$ such that $\mu[I+\varepsilon(S_w^T+S_w)+\varepsilon^2 S_w^T S_w] < e^{-2\lambda^F}$. For different scales, the critical points for local synchronization can be found as shown in Fig. 1. Then the condition for local synchronization is obtained. For the case of global synchronization, we have $\sup|f'|^2=a^2$. Applying the condition in Eq. (18) we find that the system will globally synchronize if $(\varepsilon N-1)^2<\frac{1}{a^2}$. Further more, we also validate

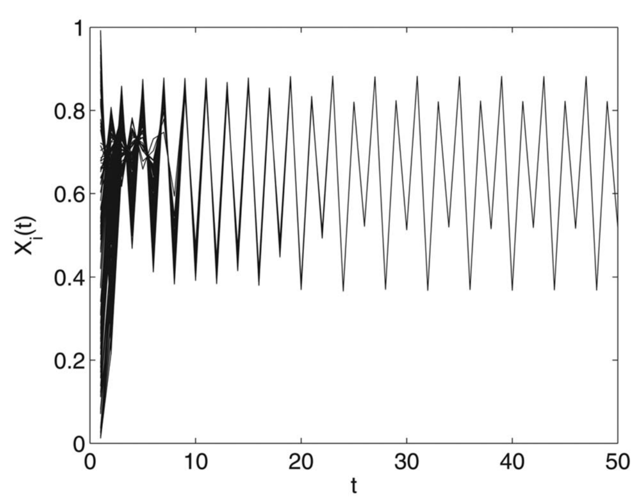

FIG. 2. The dynamic behaviors of each node in the weighted network. Here network parameters are $N=100$, $m=5$, and $\langle k\rangle=10$

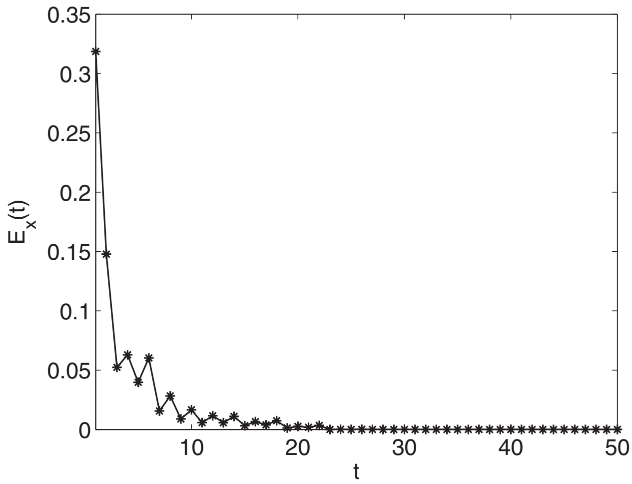

FIG. 3. Synchronization absolute errors varying with time in weighted network with $N=100$, $m=5$, and $\langle k \rangle=10$.

our results for directed weighted networks by numerical simulations. In contrast with the case of globally coupled networks, there is no general expression of the matrix measure with respect to the coupling strength. For directed weighted networks, we consider scale-free networks which are generated by using the Barabasi-Albert (BA) model [15] with $N=100$, $m=5$, and $\langle k \rangle=10$. The weight on each connection in the scale-free networks is given by $W_{ij}=1/k_i$ for all i,j.

Letting $a=3.53$ in the map, we take the coupling strength $\varepsilon=0.4$ which satisfies our criteria. From the time series of each unit in coupled system, it can be seen that all units very fast evolve in the same oscillating manner, as shown in Fig. 2. This synchronization process can also be detected by considering the average synchronization absolute error $E_X(t) = \frac{1}{N-1}\sum_{j=2}^{N}|X_{1j}(t)|$ with respect to x_1. It is shown in Fig. 3 that this system rapidly synchronizes to the reference direction.

To conclude, we study synchronization of coupled-map networks. Conditions for the local and global synchronizations are derived using matrix measure approach. Comparing with many existing synchronization conditions, the proposed criteria do not depend on the solution of the synchronous state and give less limitations on the network connections.

P.L. and Y.W. are indebted to the financial support by China Scholarship Council. M.C. thanks NSFC project (Grant No. 60804046) and Special Doctoral Fund at University by Ministry of Education (Grant No. 20070003129). J.K. thanks the supports by SFB 555 (DFG) and BIO SIM (EU).

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