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Dynamically Reconstructed Trajectories

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Abstract:

Shortness of observed time series often hinders the extraction of important information of the underlying system. In [1] we have proposed an algorithm that generates a long trajecotory, called Dynamically reconstructed Trajectory (DRT), given many short trajectories from a system. In this paper we further illustrate the validity of our algorithm to hyperchaotic systems and to systems which exhibit multistable states space.

Natural systems often exhibit a complex behavior on different time scales. Usually, these time scales of interest are either too short to be analysed using standand time series analysis techniques or so long, that it is practically impossible to assimilate a continuous data set that completely describes the long term dynamics. A classical example of the former case is transient chaotic systems [2]. The later situation is very common in many biological, physical systems and even numerical simulations owing to physical and technical limitations [3]. Furthermore, short data sets are often problematic to be handled by standard time series analysis techniques. To overcome the problem of short data sets, we have recently proposed an algorithm [1] that generates a long trajectory given an ensemble of short trajectories observed at different instances from one system. The idea stems from the ergodic theory and uses the concept of recurrence [5] of phase space trajectories to reconstruct the dynamics. The algorithm uses the local information about flow from the short segments of trajectories from a system to rebuild long dynamical replicants of the same system, called the Dynamically Reconstructed Trajectories (DRTs). The DRTs can be then studied by any data analysis techniques to obtain the desired information. In [1] we demonstrated the validity of the algorithm for three chaotic paradigmatic systems: the logistic map, the Hénon map and the Rössler oscillator. Here, we further exemplify the application of our algorithm to more complex systems, namely, the hyperchaotic Rössler system and the standard map. But first, we give below a brief description of the algorithm. Let $\vec{x}_i^j$ (where $i = 1, \ldots, N; j = 1, \ldots, K$) represent the ensemble of K short phase space trajectories of length N obtained from a series of experiments or simulations. The K short trajectories are concatenated to produce a long concatenated time series $\vec{x}_1, \ldots, \vec{x}_L$ where $L = N \cdot K$. The set of nearest neighbours of each phase space vector $(\vec{x}_i)$ is estimated for a given threshold ε, i.e., we estimate

$$\mathcal{N}_i = \{\vec{x}_j | ||\vec{x}_i - \vec{x}_j|| < \varepsilon\}$$

for $i = 1, \ldots, L$, where $\|\cdot\|$ denotes a norm, e.g., the Euclidean or the Maximum norm. The construction of the DRTs starts with the random selection of a point, $\vec{x}_a$ (where $a < L$) from the concatenated short trajectories. This constitutes the first point of the DRT. The next point of the DRT is chosen based on the condition $a \bmod N = 0$. If $a \bmod N = 0$, we are at the end of a short trajectory and $\vec{x}_{a+1}$ is not a *real* future of $\vec{x}_a$. Hence, the next point of the DRT is chosen as the future of one of its nearest neighbours with respect to the threshold ε. But if $\vec{x}_a$ is not at the end of a short segment, i.e. $a \bmod N \neq 0$, then the next point of the DRT is chosen to be either $\vec{x}_{a+1}$ or the future of one of its neighbours. In this case, the decision to continue with $\vec{x}_{a+1}$ or jump to the future of a neighbour is made with a probability p, called *inter segment jumping probability*. This procedure is repeated until a DRT of the desired length L_D is generated, where $L_D \leq L$. If during the process of the reconstruction we arrive at a point, which is at the end of a ST and also has no neighbours with respect to ε, then we restart the algorithm starting at a different initial point $\vec{x}_b$. If the algorithm fails to generate a DRT after, say, 1,000 trials, then the process is quitted assuming that it is not possible to find a DRT for the given ensemble of short trajectories and the chosen parameters p and ε. This is only an indication of the fact that the parameters ε and p of the algorithm have not been chosen appropriately or that the distribution of the ensemble of short trajectories does not sufficiently describe the underlying system. Next we exemplify the validity of the algorithm for model systems exhibiting more complex dynamics, which are closer to real world systems. The first system we analyse is the hyperchaotic Rössler oscillator [6], which has two positive Lyapunov exponents [5]. The flow of the hyperchaotic Rössler is described by the following set of equations:

$$\dot{x} = -y - z; \quad \dot{y} = x + 0.25y + w; \quad \dot{z} = 3 + xz; \quad \dot{w} = -0.5z + 0.05w. \tag{1}$$

Starting with $x_0 = -20, y_0 = 0, z_0 = 0$ and $w_0 = 15$, a long trajectory is generated using the fourth-order Runge–Kutta method and additionally, an ensemble of $K = 100$ short trajectories is generated by randomly choosing 100 different initial conditions. The length of each short trajectory is $N = 200$. Setting $p = 0.01$ and $\varepsilon = 1.0$, a DRT of length $L_D = 10,000$ is generated applying our algorithm. The generated DRT resembles the original attractor (Fig. 1(b)) very closely. Note that the jumps observed in the phase portrait of the concatenated trajectory ((Fig. 1(a)) are not present any more in the phase portrait of the generated DRT (Fig. 1(b)). Furthermore, the mutual information function (MI) [5] of the DRT is also very close to that of the long original trajectory (Fig. 1(d)). Hence, we can say that the DRT mimics the original long trajectory both qualitatively and quantitatively. To analyse the role played by the two parameters p and ε of the algorithm, the mutual information of 100 long trajectories of the hyperchaotic Rössler and and 100 DRTs (generated from the above ensemble) are compared for lags $1 \leq \tau \leq \tau_{max}$. To quantify the closeness of the DRTs to the original long trajectories we compute the error measure:

$$E_{MI} = \frac{1}{\tau_{max}} \sum_{\tau=1}^{\tau=\tau_{max}} \frac{|\mu(\tau) - \mu'(\tau)|}{0.5(\sigma(\tau) + \sigma'(\tau))},$$

where $\mu(\tau)$ is the mean value of the MI at lag τ of the ensemble of DRTs and $\mu'(\tau)$ is the mean value of the MI at lag τ of the ensemble of long trajectories. Similarly, $\sigma(\tau)$ and $\sigma'(\tau)$ are the standard deviations of the MI function at lag τ estimated from 100 DRTs and 100 long trajectories, respectively. As expected, the results show that frequent jumps, i.e. high values of p, or large jumps, i.e. high values of ε, cause the DRTs to deviate from the dynamics of the original long trajectories (Fig. 1(e)). On the other hand, if p and ε are too small, the number of nearest neighbours decreases, thereby causing repetition of segments of short trajectories. This also leads to an increase of the error E_{MI}. Hence, intermediate values of ε and p (in the region $\varepsilon \approx 3.0$ and $0 \leq p \leq 0.03$) the best choice. The second system we investigate is

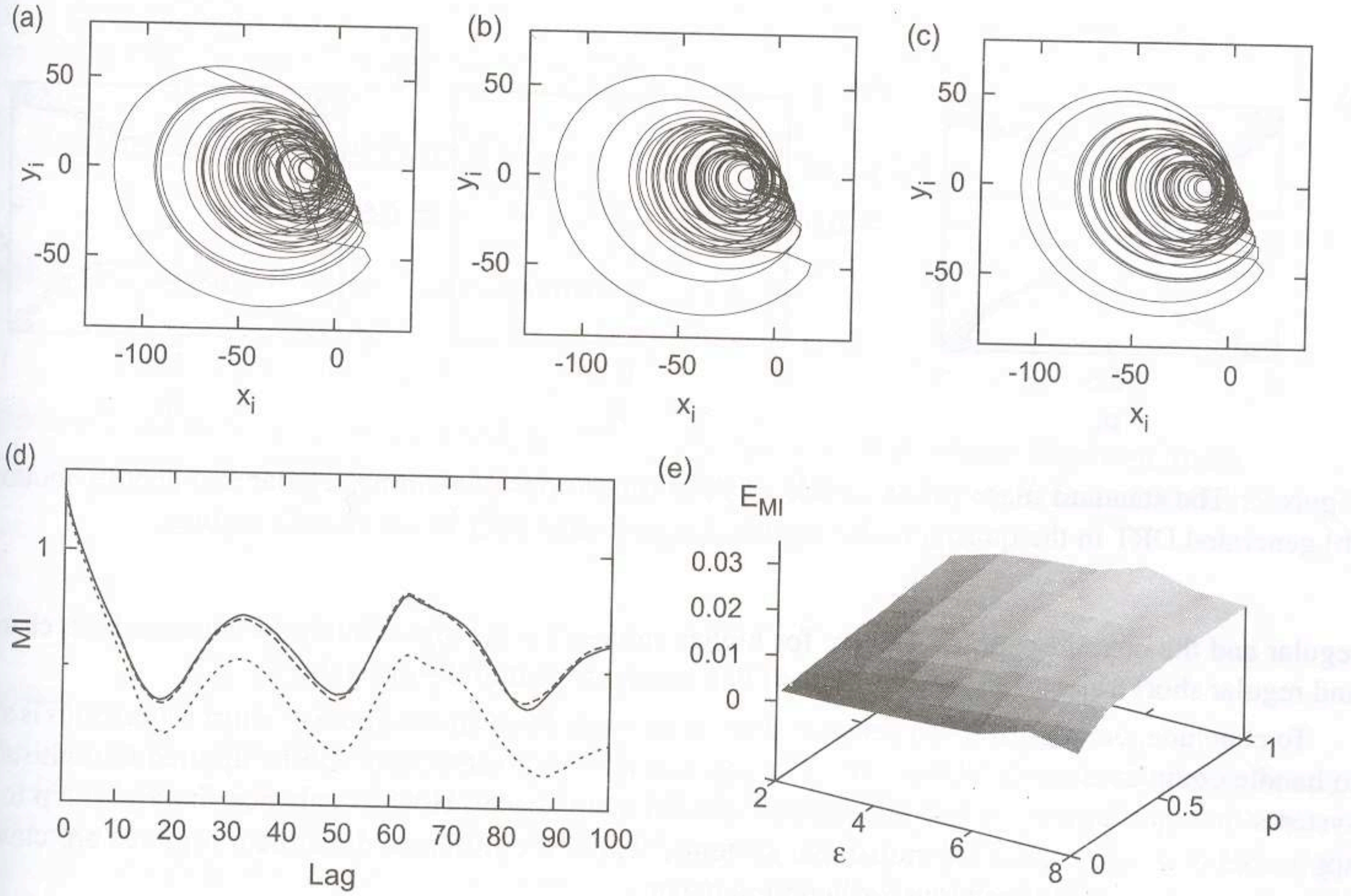

Figure 1: Phase space projections of (a) the concatenated short trajectories with $N = 200$ and $K = 100$, (b) the generated DRT, (c) a long original trajectory of the hyperchaotic Rössler. (d) Mutual information (MI): dotted line - concatenated trajectory; dashed line - DRT and solid line - long original trajectory. (e) Dependence of the error in MI on the parameters of the algorithm ε and p.

a simple chaotic map that displays Hamiltonian chaos, namely, the standard map [7]. It is given by the following equations

$$p_{i+1} = p_i + \kappa \sin(\theta_i); \quad \theta_{i+1} = \theta_i + p_{i+1}. \tag{2}$$

Depending on the value of κ and the initial conditions chosen, we can have sevaral dynamical regimes in the phase space, i.e., periodic, quasiperiodic or chaotic. For $\kappa = 0.8$ and the initial conditions (i) $p_0 = 0.0, \theta_0 = 0.6$, (ii) $p_0 = 0.0, \theta_0 = 0.1$, the system exhibits a sharply separated bistable state space, which consists of quasiperiodic and chaotic dynamics, respectively. First, we concatenate $K = 100$ short trajectories of length $N = 100$ from both dynamical regimes. A DRT of length $L_D = 5,000$ is then generated applying our algorithm with parameters $p = 0.01$ and $\varepsilon = 0.01$. Though the ensemble consists of short trajectories from two different dynamical regimes, for smaller values of ε the generated DRT is found to resort to one of the two dynamical regimes. Since the phase space is sharply divided, for smaller thresholds the nearest neighbors of regular regime will also belong to the same regime and vice versa. Hence, the dynamics of the DRT depends upon the randomly chosen first point of the DRT (Fig. 2). The resulting DRT, which is either quasiperiodic or chaotic, is found to behave quantitatively and qualitatively similar to one long quasiperiodic or chaotic trajectory, respectively. If ε becomes larger, the nearest neighbours of any point of the ensemble will be a set that consists of points from both the

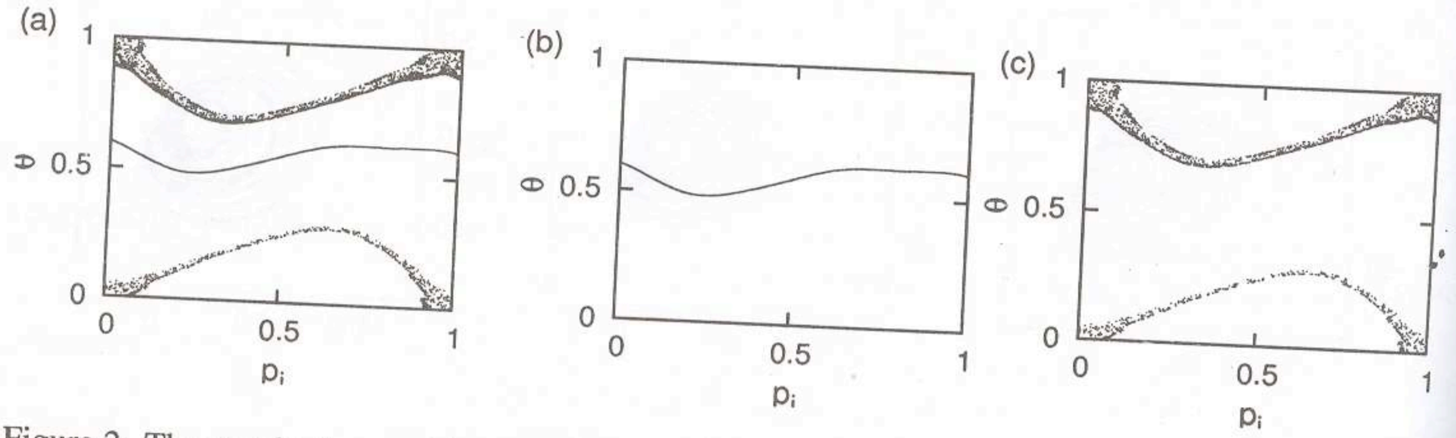

Figure 2: The standard map: (a) ensemble of short trajectories containing regular and chaotic motions, (b) generated DRT in the quasiperiodic regime, (c) generated DRT in the chaotic regime.

regular and the chaotic regions. Hence, for higher values of ε the algorithm hops between both chaotic and regular short trajectories, resulting in a DRT similar to that of the ensemble.

To conclude, our algorithm to generate long trajectories from an ensemble of short trajectories is able to handle complex dynamical systems. The algorithm in its present form can be applied to multistable systems that exhibit multistable but sharply divided state space. How to enhance the algorithm to be applicable to a wider class of mutistable systems, where the different dynamical regimes are closely embedded, will be addressed in a forthcoming paper.

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