**EPJ manuscript No.**
(will be inserted by the editor)

# Phase Synchronization in Unidirectionally Coupled Ikeda Time-delay Systems

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**Abstract.** Phase synchronization in unidirectionally coupled Ikeda time-delay systems exhibiting non-phase-coherent hyperchaotic attractors of complex topology with highly interwoven trajectories is studied. It is shown that in this set of coupled systems phase synchronization (PS) does exist in a range of the coupling strength which is preceded by a transition regime (approximate PS) and nonsynchronous regime. However, exact generalized synchronization does not seem to occur in the coupled Ikeda systems (for the range of parameters we have studied) even for large coupling strength, in contrast to our earlier studies in coupled piecewise-linear and Mackey-Glass systems [27,28]. The above transitions are characterized in terms of recurrence based indices, namely generalized autocorrelation function $P(t)$, correlation of probability of recurrence (CPR), joint probability of recurrence (JPR) and similarity of probability of recurrence (SPR). The existence of phase synchronization is also further confirmed by typical transitions in the Lyapunov exponents of the coupled Ikeda time-delay systems and also using the concept of localized sets.

## 1 Introduction

Synchronization of chaotic oscillations has been an area of extensive research since the pioneering works of Fujisaka and Yamada [1] and of Pecora and Carroll [2]. Since the identification of complete (identical) chaotic synchronization, different kinds of chaotic synchronizations have been identified and demonstrated both theoretically and experimentally (cf. [3–6]). Among the basic kinds of synchronization, chaotic phase synchronization (CPS) plays a crucial role in understanding a large class of weakly interacting nonlinear dynamical systems in diverse natural systems like cardiac and respiratory systems, biological clocks synchronized by day and night rhythms, ecological systems entrained by seasonal cycles, etc. [4,5]. The definition of CPS is a direct extension of the classical definition of synchronization of periodic oscillations and can be referred to as entrainment between the phases of interacting chaotic oscillators, while their amplitudes remain chaotic and, in general, uncorrelated [7,8].

The notion of CPS has been investigated so far in oscillators driven by external periodic forces [8,9], chaotic oscillators with different natural frequencies and/or with parameter mismatches [7,11,12], in arrays of coupled chaotic oscillators [13,14] and also in different chaotic systems [15,16]. In addition CPS has also been demonstrated experimentally in various sys-

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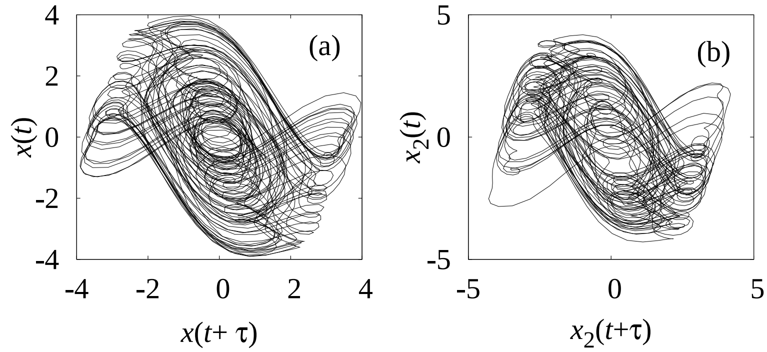


**Fig. 1.** (a) The non-phase coherent chaotic attractor of the drive, $x_1(t)$, for the value of delay time $\tau = 2$ and (b) the non-phase coherent hyperchaotic attractor of the response, $x_2(t)$, for the value of delay time $\tau = 3$ for the Ikeda system (1).

tems such as electrical circuits [15,17–19], lasers [20,21], fluids [22], biological systems [23,24], climatology [25], etc.

While the notion of CPS has been well understood in low-dimensional systems as mentioned above, it has not yet been studied in detail in nonlinear time-delay systems. These are essentially infinite-dimensional systems and correspond to an important class of dynamical systems representing several physical phenomena in diverse areas of science and technology including neuroscience, physiology, ecology, lasers, etc., [26]. Studying the nature of onset of CPS and transition to other synchronized states has received considerable attention recently due to their importance in understanding the dynamical nature of the underlying physical systems. Recently, we have reported the existence of phase, CPS and its transition to generalized synchronization (GS) in coupled time-delay systems such as piece-wise linear and Mackey-Glass time-delay systems [27,28], which typically exhibit highly non-phase-coherent chaotic and hyperchaotic attractors. We have introduced a nonlinear transformation to capture the phase of non-phase-coherent attractors of both the systems. We have also used recurrence based indices such as $P(t)$, CPR, JPR and SPR directly to the non-phase-coherent attractors and typical transitions in the Lyapunov exponents of the coupled time-delay systems to characterize the synchronization transitions. We have also found that all the three approaches are in good agreement in indicating the onset of CPS and its transitions.

In this paper, as a natural extension of our above investigations, we try to generalize the nonlinear transformation and to test the validity of the above mentioned recurrence based indices in identifying the synchronization transitions in general class of time-delay systems exhibiting highly non-phase-coherent hyperchaotic attractors of more complex topology. As an example, we have considered one of the prototype time-delay systems, namely Ikeda time-delay system [29], which exhibits highly non-phase-coherent hyperchaotic attractor with complex topology for suitable parameter values. Even though, we have not yet succeeded in generalizing the nonlinear transformation to capture the phase of the non-phase-coherent hyperchaotic attractors of the Ikeda system, we found that the recurrence based indices serve as excellent quantifiers in identifying the transition from non-synchronized to phase synchronized state both qualitatively and quantitatively in the coupled Ikeda systems. We have also characterized these transitions by typical changes in the Lyapunov spectrum of the coupled Ikeda time-delay systems. Further, we have confirmed the existence of CPS using the concept of localized sets.

The plan of the paper is as follows. In Sec. II, we briefly point out the inadequacy of the conventional methods available in the literature in identifying phase in time-delay systems and the necessity of specialized tools and techniques to identify phase in such systems, while in Sec. III we discuss briefly about the Ikdea time-delay systems and its dynamics. We demonstrate the onset of CPS and its transition to CPS in coupled Ikeda systems using recurrence based indices and Lyapunov exponents in Sec. IV. Finally in Sec. V, we summarize our results.

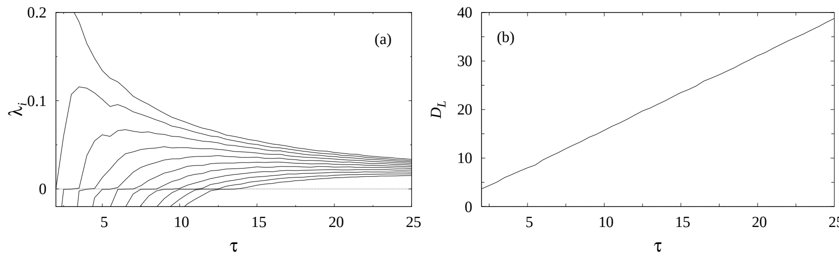


**Fig. 2.** (a) Spectrum of the first eleven largest Lyapunov exponents for the values of the parameters $a = 1, b = 5$ in the range of delay time $\tau \in (2, 25)$ and (b) Kaplan-Yorke dimension $D_L$ in the corresponding range of delay time for the Ikeda system (1).

## 2 CPS and Time-delay systems

As noted in the introduction, CPS has been studied extensively during the last decade in various nonlinear dynamical systems. However, only a few methods are available in the literature [4,5] to calculate the phase of chaotic attractors. Unfortunately most of these measures are restricted to phase-coherent chaotic attractors, while a few of them are applicable to non-phase-coherent chaotic attractors of low-dimensional systems as well. However, all these conventional methods which are applicable to the phase-coherent and non-phase-coherent attractors cannot be used in the case of time-delay systems in general, because such systems very often exhibit more complicated attractors with more than one positive Lyapunov exponents. Correspondingly methods to calculate the phase of non-phase-coherent hyperchaotic attractors of time-delay systems are not available. The most promising approach available in the literature to calculate the phase of non-phase-coherent attractors is based on the concept of curvature [30], but this is often restricted to low-dimensional systems and it does not work in the case of nonlinear time-delay systems in general, where very often the attractor is non-phase-coherent and high-dimensional. This is essentially be due to the multiple intrinsic characteristic time scales of the nonlinear time-delay systems. Hence defining and estimating phase from the hyperchaotic attractors of the time-delay systems itself is a challenging task and so specialized techniques and tools have to be identified to introduce the notion of phase in such systems.

Recently, the present authors have studied in some detail the existence of phase and CPS in time delay systems admitting non-phase-coherent hyperchoatic attractors and specifically analyzed coupled piecewise linear and couped Mackey-Glass systems. Three different approaches were introduced to identify and calculate phase and consequently CPS between the interacting time-delay systems: 1) identifying suitable nonlinear transformation which can unfold the complicated chaotic and hyperchaotic attractors with multiple loops into smeared limit cycle like attractor, 2) using recurrence based indices such as generalized autocorrelation function $P(t)$, correlation of probability of recurrence (CPR), joint probability of recurrence (JPR) and similarity of probability of recurrence (SPR) directly to the non-phase-coherent chaotic and hyperchaotic attractors, we have demonstrated the existence of CPS both qualitatively and quantitatively and 3) finally the onset of CPS and their transition is also characterized by typical transitions in the spectrum of Lyapunov exponents of the coupled time-delay systems. We now apply these approaches to coupled Ikeda systems which exhibit even more complicated chaotic and hyperchaotic non-phase-coherent attractors with complex topological properties having highly interwoven trajectories and identify the nature of CPS. We have also confirmed the existence of CPS using the framework of localized sets.

## 3 The Ikeda time-delay system

The Ikeda system was introduced to describe the dynamics of an optical bistable resonator and it was shown that the transmitted light from a ring cavity containing a nonlinear dielectric medium undergoes a transition from a stationary state to periodic and nonperiodic states, when the intensity of the incident light is increased. It has also been shown that the nonperiodic state is characterized by a chaotic variation of the light intensity and associated broadband noise in the power spectrum [29]. Ikeda system is well known for delay induced chaotic behavior [32–34] and this system has also been receiving focus on synchronization studies recently [35–38]. The Ikeda model is specified by the state equation

$$\dot{x} = -ax(t) - b \sin x(t - \tau), \tag{1}$$

where $a > 0$ and $b > 0$ are the parameters and $\tau$ is the delay time. Physically $x(t)$ is the phase lag of the electric field across the resonator and thus may clearly assume both positive and negative values, $\alpha$ is the relaxation coefficient, $b$ is the laser intensity injected into the system and $\tau$ is the round-trip time of the light in the resonator. Typical chaotic and hyperchaotic chaotic attractors of the Ikeda system are shown in Figs. 1a and b for values of delay times $\tau = 2$ and $\tau = 3$, respectively, while the other parameter values are fixed as $a = 1.0$ and $b = 5$. The first eleven largest Lyapunov exponents of the Ikeda system for the parameters $a = 1.0, b = 5$ in the range of delay time $\tau \in (2, 25)$ are shown in Fig. 2a and the corresponding Kaplan-Yorke Lyapunov dimension calculated using the formula

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}, \tag{2}$$

where j is the largest integer for which $\lambda_1 + ... + \lambda_j \geq 0$, is shown in Fig. 2b.

## 4 CPS in coupled Ikeda time-delay systems

We consider the following unidirectionally coupled drive $x_1(t)$ and response $x_2(t)$ systems

$$\dot{x}_1(t) = -ax_1(t) + b_1 \sin x_1(t - \tau_1), \tag{3}$$

$$\dot{x}_2(t) = -ax_2(t) + b_2 \sin x_2(t - \tau_2) + b_3 \sin x_1(t - \tau_1), \tag{4}$$

where the parameters are fixed as $a = 1.0, b_1 = b_2 = 5.0$. The delay times $\tau_1 = 2$ and $\tau_2 = 3$ provide parameter mismatch between the drive, $x_1(t)$, and the response, $x_2(t)$, systems and $b_3$ is the coupling strength. In the absence of the coupling both systems evolve independently and the attractor of the drive system shown in Fig. 1a is chaotic for the value of delay time $\tau = 2$ and that of the response system shown in Fig. 1b is hyperchaotic with two positive Lyapunov exponents for the value of delay time $\tau = 3$ as evidenced from the spectrum of Lyapunov exponents shown in Fig. 2a. Hence both systems are qualitatively different and their attractors shown in Figs. 1 are highly non-phase-coherent with interwoven trajectories exhibiting complex topological properties. When the coupling strength $b_3$ is increased from zero, the degree of chaotic phase synchronization between the drive and the response systems increases after certain threshold value of the coupling strength and finally they become phase synchronized fully. However, further increase in the coupling strength does not lead to a transition to generalized synchronization even for appreciably larger value of $b_3$ unlike the case of coupled piecewise linear delay systems or coupled Mackey-Glass systems [27,28]. Now these results are depicted using recurrence based indices, namely, $P(t)$, CPR, JPR and SPR.

### 4.1 Recurrence based indices

Synchronization transition in coupled Ikeda systems (3) and (4), that is from desynchronized state to phase synchronized state and then possibly to generalized synchronized state, can be analyzed by means of recurrence based indices even when the corresponding attractors have complex topological properties. The generalized autocorrelation function $P(t)$ has been introduced in Refs. [31,39] as

$$P(t) = \frac{1}{N-t} \sum_{i=1}^{N-t} \Theta(\epsilon - ||X_i - X_{i+t}||), \tag{5}$$

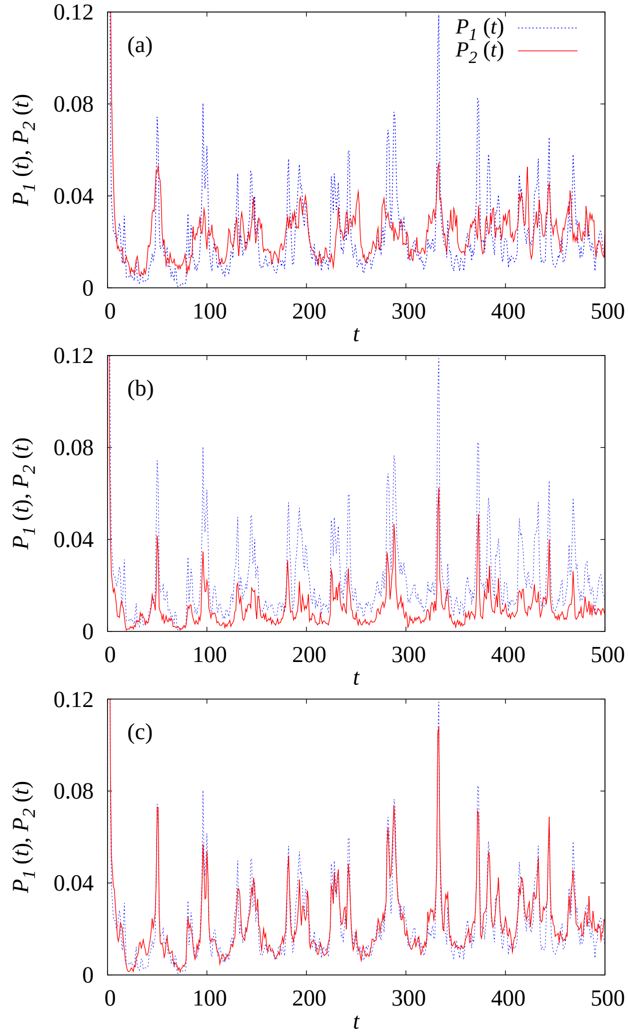


**Fig. 3.** (Color online) Generalized autocorrelation functions of both the drive $P_1(t)$ and the response $P_2(t)$ systems. (a) Non-synchronization for $b_3 = 4$, (b) Approximate phase synchronization for $b_3 = 10$ and (c) Phase synchronization for $b_3 = 20$.

where $\Theta$ is the Heaviside function, $X_i$ is the $i$th data corresponding to either the drive variable, $x_1$, or the response variable, $x_2$, and $\epsilon$ is a predefined threshold. $||.||$ is the Euclidean norm and $N$ is the number of data points. $P(t)$ can be considered as a statistical measure about how often the phase $\phi$ has increased by $2\pi$ or multiples of $2\pi$ within the time $t$ in the original space. If two systems are in CPS, their phases increase on average by $K \cdot 2\pi$, where $K$ is a natural number, within the same time interval $t$. The value of $K$ corresponds to the number of cycles when $||X(t+T) - X(t)|| \sim 0$, or equivalently when $||X(t+T) - X(t)|| < \epsilon$, where $T$ is the period of the system. Hence, looking at the coincidence of the positions of the maxima of $P(t)$ for both the systems (3) and (4), one can qualitatively identify CPS.

A criterion to quantify CPS is the cross correlation coefficient between the drive, $P_1(t)$, and the response, $P_2(t)$, which can be defined as Correlation of Probability of Recurrence (CPR),

$$CPR = \langle \bar{P}_1(t)\bar{P}_2(t)\rangle/\sigma_1\sigma_2, \tag{6}$$

where $\bar{P}_{1,2}$ means that the mean value has been subtracted and $\sigma_{1,2}$ are the standard deviations of $P_1(t)$ and $P_2(t)$, respectively. If the two systems (3) and (4) are in CPS, the probability of

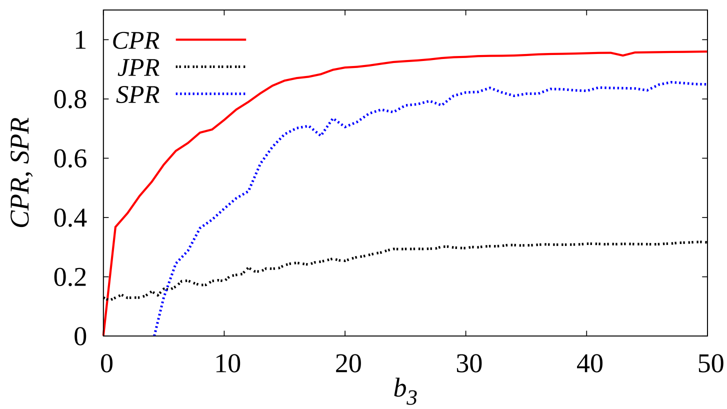


**Fig. 4.** (Color online) Indices CPR, SPR as a function of coupling strength $b_3 \in (0, 50)$.

recurrence is maximal at the same time $t$ and CPR $\approx 1$. If they are not in CPS, the maxima do not occur simultaneously and hence one can expect a drift in the probability of recurrences which results in low values of CPR.

When the coupled Ikeda systems (3) and (4) are in generalized synchronization, two close states in the phase space of the drive variable correspond to that of the response. Hence the neighborhood identity is preserved in phase space. Since the recurrence plots are nothing but a record of the neighborhood of each point in the phase space, one can expect that their respective recurrence plots are almost identical. Based on these facts the following two indices can be calculated as proposed in [39] to quantify GS for the Ikeda systems similar to the coupled piecewise linear and Mackey-Glass systems analyzed by us recently [27,28].

First, the authors of [39] proposed the Joint Probability of Recurrences (JPR),

$$JPR = \frac{\frac{1}{N^2}\sum_{i,j}^{N} \Theta(\epsilon_x - ||X_i - X_j||)\Theta(\epsilon_y - ||Y_i - Y_j||) - RR}{1 - RR} \qquad (7)$$

where $RR$ is rate of recurrence, $\epsilon_x$ and $\epsilon_y$ are thresholds corresponding to the drive and response systems, respectively such that $RR_X = RR_Y = RR$ and $X_i$ is the $i$th data corresponding to the drive variable $x_1$ and $Y_i$ is the $i$th data corresponding to the response variable $x_2$. RR measures the density of recurrence points and it is fixed as 0.02 [39]. JPR is close to 1 for systems in GS and is small when they are not in GS. The second index depends on the coincidence of the probability of recurrence, which is defined as Similarity of Probability of Recurrence (SPR),

$$SPR = 1 - \langle (\bar{P}_1(t) - \bar{P}_2(t))^2 \rangle / \sigma_1 \sigma_2. \qquad (8)$$

SPR is again of order 1 if the two systems are in GS and approximately zero or negative if they evolve independently.

Now, we will apply these concepts to the original non-phase-coherent attractors shown in Figs. 1, when coupling is introduced as in Eqs. (3) and (4). We estimate these recurrence based measures from 5000 data points after barring out sufficient transients with the integration step $h = 0.01$ and sampling rate $\Delta t = 100$. The generalized autocorrelation functions $P_1(t)$ of the drive $x_1(t)$ system and $P_2(t)$ of the response $x_2(t)$ systems are depicted in Figs. 3 for different values of the coupling strength. The maxima of the generalized autocorrelation functions $P_1(t)$ and $P_2(t)$ do not occur simultaneously (Fig. 3a) and there exists a drift between them for the value of the coupling strength $b_3 = 4$ and hence both the systems evolve independently. This fact is also reflected in the rather low values of the indices CPR, JPR and SPR as shown in Fig. 4. Looking into the details of the generalized autocorrelation functions in Fig. 3b for the value of the coupling strength $b_3 = 10$, we find that the main oscillatory dynamics becomes locked and hence the large amplitude peaks (maxima) of $P_1(t)$ and $P_2(t)$ coincide while small amplitude peaks do not. This behavior is observed in the range of $b_3 \in (4.2, 20)$, which corresponds to

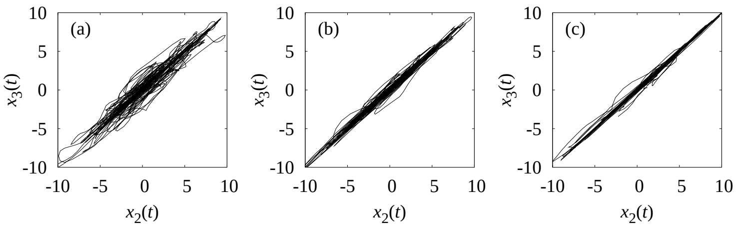


**Fig. 5.** The plot of response variable $x_2(t)$ Vs auxiliary variable $x_3(t)$ for the values of the coupling strengths (a) $b_3 = 30$, (b) $b_3 = 40$ and (c) $b_3 = 50$.

the transition regime (approximate CPS) and this is also indicated by a smooth increase in the value of CPR (Fig. 4) towards the value 1.

Further increase in the value of the coupling strength beyond $b_3 = 20$ results in an almost perfect locking of all the oscillatory dynamics of the coupled system. Consequently, a majority of the positions of the peaks in the generalized autocorrelation functions $P_1(t)$ and $P_2(t)$ agree with each other as illustrated in Fig. 3c for the value of $b_3 = 20$. However, it is observed that the magnitude of peaks are generally of different values and this difference in the heights of peaks indicates that there is no correlation in the amplitudes of both the systems. This is in accordance with strongly bounded nature of phase difference and further increase in the value of the coupling strength results in a saturation in the value of CPR $\approx 1$ as seen from the Fig. 4, which is a strong indication for the existence of CPS.

Even for a very large value of the coupling strength, say $b_3 = 50$, the amplitudes of the maxima in the generalized autocorrelation function do not coincide and hence one does not find an indication towards the exact GS. This is further confirmed from the rather lower values of JPR and SPR depicted in Fig. 4. This scenario is in contrast to our earlier studies [27, 28] where there exists transition from phase to generalized synchronization within reasonable range of values of the coupling strength in the case of piece-wise linear and Mackey-Glass time-delay systems. This is further confirmed from the auxiliary system approach by augmenting the coupled Ikeda systems ((3) and (4)) with an additional auxiliary system for the variable $x_3(t)$ identical to the response system, satisfying the equation

$$\dot{x}_3(t) = -ax_3(t) + b_2 \sin x_3(t - \tau_2) + b_3 \sin x_1(t - \tau_1). \quad (9)$$

We have analyzed numerically the combined system of equations (3), (4) and (9). The plot of the response variable $x_2(t)$ Vs the auxiliary variable $x_3(t)$ for the values of the coupling strengths $b_3 = 30, 40$ and 50 are depicted in Figs. 5a, 5b and 5c, respectively. As may be noted that for none of these values one obtains a sharp diagonal line, indicating only the existence of approximate GS. A possible reason for this is that the largest Lyapunov exponents of the response system do not attain negative saturation even for larger values of the coupling strength as shown in Fig. 6. Hence there does not seem to exist exact generalized synchronization between the coupled Ikeda systems in the explored range of parameters by us, even though CPS does exist in this range.

### 4.2 Spectrum of Lyapunov exponents

The transition from non-synchronization to CPS is also characterized by changes in the spectrum of Lyapunov exponents of the coupled time-delay systems (3) and (4). The spectrum of the first five largest Lyapunov exponents of the coupled Ikeda systems is shown in Fig. 6. The null Lyapunov exponent of the response system $x_2(t)$ becomes negative at the value of the coupling strength $b_3 = 4.2$ while the other two largest Lyapunov exponents remain positive, which is a typical characteristic feature for the onset of CPS in the coupled systems. The second least positive Lyapunov exponent becomes negative at the value of the coupling strength $b_3 = 13.5$, an indication of onset of correlation in amplitudes of both the interacting dynamical systems,

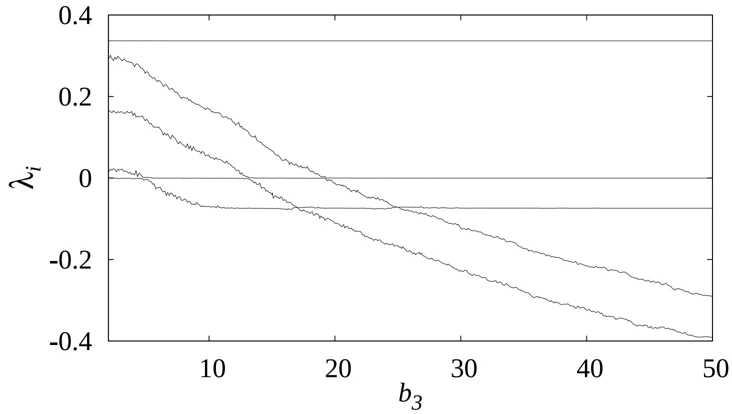


**Fig. 6.** Spectrum of first five largest Lyapunov exponents of the coupled Ikeda systems (3) as a function of coupling strength $b_3 \in (0, 50)$.

while the largest positive Lyapunov exponent of the response system becomes negative at the value of $b_3 = 20$. This is a strong indication that in this rather complex attractor the amplitudes become somewhat interrelated already at the transition to PS in the range $b_3 \in (4.2, 20)$ in agreement with our earlier results in coupled Mackey-Glass systems and (as in the funnel attractor [30]). However, it is observed that there does not exist generalized synchronization between the coupled Ikeda systems even for larger values of the coupling strength in contrast to our earlier studies [27,28] and this is evident from the value of the Lyapunov exponents of the response system becoming increasingly negative without attaining saturation.

### 4.3 Concept of Localized Sets

Recently, an interesting framework for identifying phase synchronization without having explicitly the measure of the phase, namely the concept of localized sets, has been introduced [40]. The basic idea of this concept is that one has to define a typical event in one of the coupled oscillators and then observe the other oscillator whenever this event occurs. These observations give rise to a set $D$. Depending upon the property of this set $D$ one can state whether there exists PS or not. The coupled oscillators evolve independently if the sets obtained by observing the corresponding events in both the oscillators spread over the attractors of the oscillators. On the other hand, if the sets are localized on the attractors then PS exist between the interacting oscillators.

We have confirmed the existence of CPS in the coupled Ikeda time-delay systems also by using the concept of localized set. We have defined the event in the attractor of the drive system as a segment characterized by $x_1(t+\tau) = 0$ and $x_1(t) > 2.0$ and another event in the response system as a segment characterized by $x_2(t+\tau) = 0$ and $x_2(t) < -3.0$, which are shown as black lines in Fig. 7. The sets obtained by observing the response Ikeda system whenever the defined event occurs in the drive system and vice versa are shown as dots in Figs. 7a and 7b, respectively, for the value of the coupling strength $b_3 = 4.0$, for which there is no CPS as discussed earlier and hence the sets are spread over the attractors. On the other hand for the value of the coupling strength $b_3 = 20$ for which CPS exists as seen from Figs. 3-6, the sets are localized as shown in Figs. 7c and 7d confirming the existence of CPS in the coupled Ikeda systems.

## 5 Summary and Conclusion

We have identified the existence of CPS in coupled Ikeda time-delay systems which possess highly non-phase-coherent chaotic and hyperchaotic attractors with complex topology. In par-

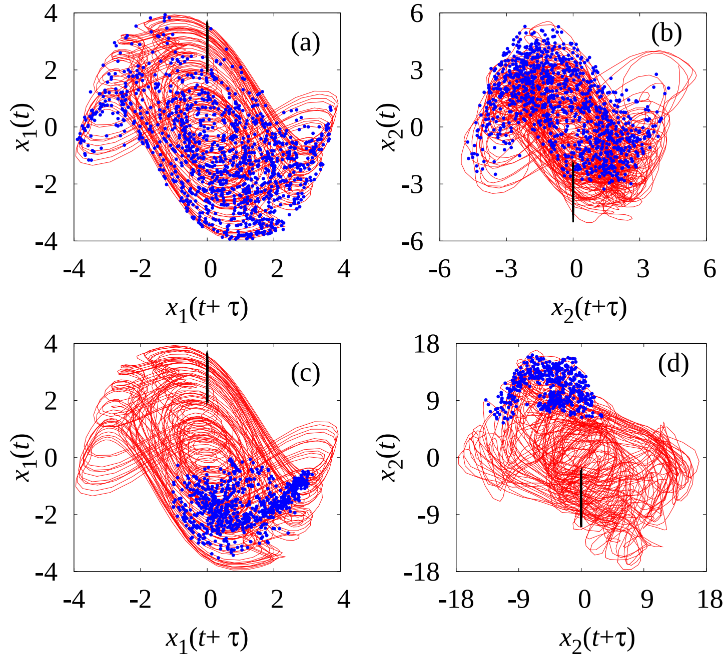


**Fig. 7.** (Color online) (a) and (c) attractor of the drive system, (b) and (d) attractor of the response system. The bars indicate the events in the corresponding attractors. In (a) and (b) the sets spread over the attractor and hence there is no CPS for the value of the coupling strength $b_3 = 4.0$ and, in (c) and (d) the sets are localized confirming the existence of CPS for $b_3 = 20.0$.

ticular, we have shown that there is a typical transition from a nonsynchronized state to CPS as a function of the coupling strength. We have characterized this transition in terms of recurrence based indices such as $P(t)$, CPR, JPR and SPR, and quantified the different synchronization regimes in terms of them. The transition is also confirmed by the typical transition in the Lyapunov exponents of the coupled Ikeda time-delay systems. Further, we have also confirmed the existence of CPS using the concept of localized sets. We have found that the recurrence based techniques are more efficient than the other conventional techniques available in the literature to identify CPS in higher dimensional systems, in particular in time-delay systems. It is also of interest to find out a suitable general transformation to include the attractors of large class of time-delay systems, including coupled Ikeda systems, which transforms the non-phase-coherent attractors into smeared limit cycle like attractors. Work is in progress on this aspect.

The work of D. V. S. and M. L. has been supported by a Department of Science and Technology, Government of India sponsored research project. The work of M. L. is supported by a DST Ramanna Fellowship. J. K has been supported by his Humboldt-CSIR research award and NoE BIOSIM (EU) Contract No. LSHB-CT-2004-005137.

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