CHAOS **18**, 037123 (2008)

Bifurcational mechanisms of synchronization of a resonant limit cycle on a two-dimensional torus

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(Received 18 February 2008; accepted 4 June 2008; published online 22 September 2008)

We study synchronization of a resonant limit cycle on a two-dimensional torus with an external harmonic signal. The regime of the resonant limit cycle is realized in a system of two coupled Van der Pol oscillators; we consider the resonances 1:1 and 1:3. We analyze the influence of coupling strength between the oscillators. We show that the resonant limit cycle can be generally synchronized on the torus through the resonance destruction followed by the locking of one and then another one of the basic frequencies. We consider the bifurcational mechanism of the synchronization effect. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2949929]

Quasiperiodic oscillations represent stable solutions of dynamical systems that depend on a finite number of periodic functions $\phi_k(\omega_k t)$, $k=1,2,\ldots,n$, with the period $T_k=2\pi/\omega_k$ in each argument. Quasiperiodic solutions describe complex oscillating processes with several frequencies that are generally independent natural frequencies of interacting subsystems. In phase space these solutions are represented by a stable limit set (attractor) in the form of an n-dimensional torus. The structure of phase trajectories on a torus can be ergodic (nonperiodic) or resonant (periodic). According to the Arnold terminology, the second type characterizes a resonant limit cycle on a torus, resulting from mutual frequency locking within Arnold's "tongues." The latter denotes synchronization regions of oscillations on a torus. In our paper the central interest is addressed to the study of synchronization of quasiperiodic oscillations by an external harmonic signal. As the simplest and most real object for consideration we choose two coupled Van der Pol oscillators with basic frequencies mismatch. The peculiarities of synchronization of two-frequency oscillations are studied in detail. The bifurcational mechanisms of synchronization are described, and the bifurcation diagram is constructed. The synchronization effects are studied for two values of the winding number, i.e., $\Theta=1:1$ and $\Theta=1:3$, and the influence of mutual coupling degree between the oscillators is analyzed. The results of the paper are in good accord with the numerical and experimental data obtained earlier for other dynamical systems.

I. INTRODUCTION

Research of peculiarities and properties of quasiperiodic oscillations compose a sufficiently complex problem.[1–5] One of the important questions in that branch is the analysis of synchronization phenomena.

The problem of synchronization of quasiperiodic motions with two independent frequencies was partly solved first in.[6,7] In Ref. 6, the winding number locking phenomenon was first investigated in a system of two coupled oscillators of two-frequency motion. It has been established that the regime of complete synchronization results from the consecutive locking of the basic frequencies of the system. In Ref. 7, the synchronization of two-frequency resonant motions by an external harmonic signal was considered. It has been shown that in the case of the synchronization of a resonant cycle on a torus, the basic frequencies are synchronized independently, although the resonant conditions assume the frequencies to be rationally related. This effect was demonstrated numerically and experimentally for the resonances $\Theta=1:4$ and $\Theta=1:3$.[7]

In our paper we study the bifurcational mechanisms of external synchronization of quasiperiodic oscillations by using the model of two coupled Van der Pol oscillators with frequency mismatch. The choice of that model is substantiated as follows. First, the oscillator of two-frequency oscillations considered in Ref. 7 provides modulated self-sustained oscillations with frequencies f_0 and f_1. Their amplitudes are significantly different and cannot be varied in necessary ranges. Second, the system used in Ref. 7 cannot realize the regime of resonance 1:1, which is quite important for understanding the bifurcational mechanisms of synchronization. Finally, it is impossible to vary the internal coupling between the subsystems of the oscillator considered in Ref. 7, and this makes the observed effects very hard to study.

Taking into account the aforementioned, we propose and consider a different model that is more convenient and does not have the above disadvantages. We deal with a system of two coupled Van der Pol oscillators, where amplitudes, frequencies, and coupling strength are determined by the corresponding parameter values. Using that model we have confirmed the results obtained in Refs. 6 and 7 as well as have

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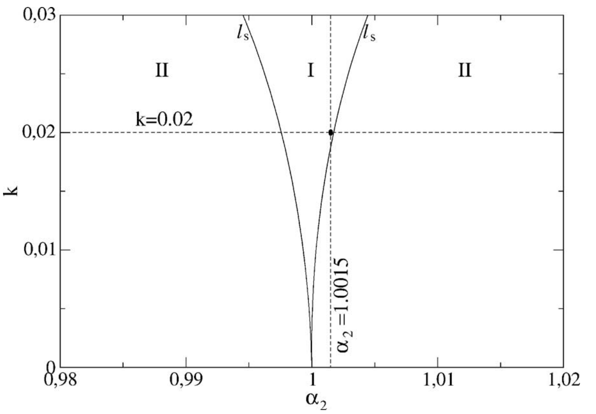

FIG. 1. Region of mutual synchronization of oscillators (2) for $m=0.1$, $\alpha_1=1$. "I" denotes the region of the resonant limit cycle existence with winding number $\Theta=1:1$, and "II" is the region of quasiperiodic motions.

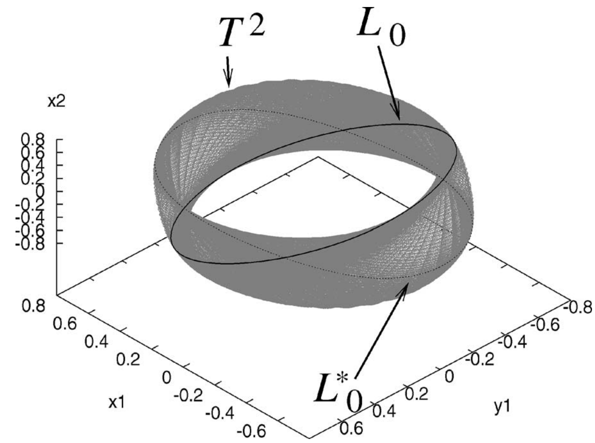

FIG. 2. Three-dimensional projection of the two-dimensional torus T^2 outside the resonant region (gray), L_0 is the stable cycle and L_0^* is the saddle cycle on the torus inside the resonant region.

determined the bifurcational mechanism of synchronization of a resonant limit cycle on a two-dimensional torus.

II. MUTUAL SYNCHRONIZATION OF TWO COUPLED OSCILLATORS

Let us first consider the classical mechanism of synchronization of two coupled periodic oscillators. As a partial self-sustained oscillatory system we analyze the model of the Van der Pol oscillator in the regime of a limit cycle, corresponding to stable nearly harmonic motions:

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = (m - x_1^2)y_1 - \alpha_1^2 x_1, \tag{1}$$

where m is the excitation parameter, $\alpha_1^2=(2\pi f_1)^2$, $f_1=1/T_0$, f_1 is the frequency, and T_0 is the period of motions. As is well known, self-sustained oscillations in system (1) appear through the Hopf bifurcation at the point $m^*=0$. The amplitude of motions for $m>m^*$ is proportional to $\sqrt{m}$.

As the second subsystem we consider the same Van der Pol oscillator (1) with a frequency mismatch ($\alpha_2 \neq \alpha_1$). We study the regime of self-sustained oscillations in the system with symmetrical coupling:

$$\begin{aligned} \dot{x}_1 &= y_1, \quad \dot{y}_1 = (m - x_1^2)y_1 - \alpha_1^2 x_1 + k(x_2 - x_1), \\ \dot{x}_2 &= y_2, \quad \dot{y}_2 = (m - x_2^2)y_2 - \alpha_2^2 x_2 + k(x_1 - x_2). \end{aligned} \tag{2}$$

Here, k determines the coupling strength between the partial oscillators, m has the same value for both oscillators, and the natural frequencies α_1 and α_2 of the oscillators have different but rather close values.

We consider self-sustained oscillations in system (2) for the following parameter values: $m=0.1$, $\alpha_1=1$, and $k=0.02$. We vary the parameter α_2 in the range $0.98<\alpha_2<1.02$ to analyze the influence of partial frequencies f_1 and f_2 mismatch on the system dynamics.

Figure 1 shows the region of synchronization that corresponds to the frequency locking at the basic tone. Inside the synchronization region (region I in Fig. 1), the first oscillator ($\alpha_1=1$) locks the frequency of the second one, and the frequencies of the interacting oscillators become equal: $f_1=f_2$. With this, in the region I the frequencies f_2 and f_1 are not generally equal to the partial frequencies of uncoupled oscillators (for $k=0$). The region I plotted on the parameter plane "coupling (k) versus mismatch (α_2)" is called an Arnold's "tongue" with the Poincaré winding number $\Theta=1:1$, and corresponds to the synchronization at the basic tone.

Outside the region of synchronization (regions II in Fig. 1), regimes of two-frequency motions are observed. Within these regions the frequencies of the subsystems are not equal ($f_1 \neq f_2$).

The next step is to consider the discussed effect from the viewpoint of the qualitative theory of differential equations.

Two-frequency quasiperiodic motions are observed in the region II, which generally corresponds to the existence of an ergodic two-dimensional torus. When one enters the region I from the region II [by crossing the bifurcational lines l_s (Fig. 1)], a new structure appears on the surface of the two-dimensional torus; namely, stable and saddle cycles. The stable cycle corresponds to the regime of mutual synchronization of two oscillators. The stable periodic motions with frequencies $f_1=f_2$ are observed in the regime of locked frequencies.

That effect is illustrated in Fig. 2, in which the projection of the two-dimensional torus T^2 and the projections of stable (L_0) and saddle (L_0^*) resonant cycles on the surface of torus T^2 are shown. It is significant that the torus T^2 exists in both region II and region I! However, in the physical experiment the stable limit cycle L_0 can be only observed in the region I because of the resonance on the torus T^2. This can be demonstrated by analyzing the Poincaré section with the hyperplane $x_1=0$ for the regimes shown in Fig. 2. The calculation results are presented in Fig. 3, where the closed curve l corresponds to the torus T^2 and points P and Q correspond to stable L_0 and saddle L_0^* cycles. Unstable separatrixes of the saddle Q come to the stable node P and form the closed invariant curve. That curve is an image of the two-dimensional torus inside the synchronization region. When crossing the bifurcational lines l_s (Fig. 1) from region I into region II, the saddle Q and the node P merge and disappear through a saddle-node bifurcation. Thus, the saddle and stable cycles exist in the region of synchronization, and the

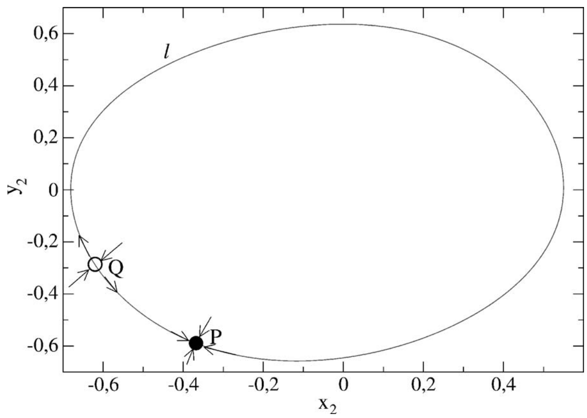

FIG. 3. Poincaré sections of the ergodic torus T^2 (closed curve l) and resonant cycles L_0 and L_0^* (points P and Q) on it that corresponds to Fig. 2.

saddle-node bifurcation of these cycles destroys the synchronization regime.

According to the aforesaid, we can draw the following conclusions that are necessary for understanding the results presented further in the current paper:

(1) The stable limit cycle lying on the surface of the two-dimensional torus corresponds to the synchronization regime.
(2) That limit cycle is not an image of motions realized in the subsystems, but results from interaction of the partial oscillators.

In conclusion, it should be specified that the above-discussed qualitative theory of mutual synchronization of two coupled oscillators also describes the regime of external synchronization. The only difference is that the frequency of synchronous (locked) motions in case of external synchronization is always equal to the frequency of the external force. Additionally, the described mechanism of synchronization at the basic tone ($\Theta=1:1$) is qualitatively the same as for $\Theta=p:q$, where p and q are rational quantities. In that case saddle-node bifurcations take place for more complicated multifold (loop) cycles, but the principle remains the same.

III. INFLUENCE OF EXTERNAL PERIODIC FORCE ON THE RESONANT LIMIT CYCLE

Now we study the influence of a periodic signal on system (2). We add the periodic perturbation $k_e \sin[(2\pi f_e)t]$ to the second equation of system (2):

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = (m - x_1^2)y_1 - \alpha_1^2 x_1 + k(x_2 - x_1) + k_e \sin[(2\pi f_e)t],$$
$$\dot{x}_2 = y_2, \quad \dot{y}_2 = (m - x_2^2)y_2 - \alpha_2^2 x_2 + k(x_1 - x_2). \tag{3}$$

We consider the regime of motions in the system (3) for $k_e=0$ that corresponds to region I. The parameter values are set as follows: $m=0.1$, $\alpha_1=1$, $\alpha_2=1.0015$, and $k=0.02$ (the black dot in Fig. 1). The autonomous system generates stable periodic motions that corresponds to the existence of stable limit cycle L_0. In physical experiments one can observe an ordinary stable periodic motion with the frequency f_1, and the power spectrum consists of odd harmonics $f_n=(2n-1)f_1$ $(n=1,2\ldots)$ by virtue of symmetry. Our detailed studies have shown that the external synchronization of a resonant limit cycle lying on the surface of a two-dimensional torus can occur in a very different way than in case of a nonresonant limit cycle.

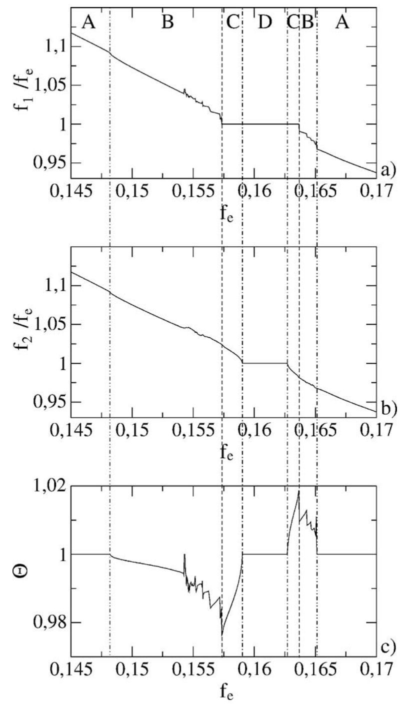

FIG. 4. Frequency relations f_1/f_e (a), f_2/f_e (b), and winding number $\Theta=f_1:f_2$ (c) as functions of the external force frequency f_e for $k_e=0.025$.

In Figs. 4(a) and 4(b), the frequency relations f_1/f_e (a) and f_2/f_e (b) are shown as functions of the external force frequency f_e for $k_e=0.025$. A dependence of the winding number $\Theta=f_1:f_2$ on the external force frequency is shown in Fig. 4(c). One can distinguish four regions A, B, C, and D, in which the system dynamics is qualitatively different. They can be appropriately characterized by the spectrum of Lyapunov exponents (Fig. 5).

In region A, the external force frequency value is rather far from the frequency of the limit cycle $f_1=f_2\approx 0.158$. Quasiperiodic motions with frequencies f_e and $f_1=f_2$ are realized

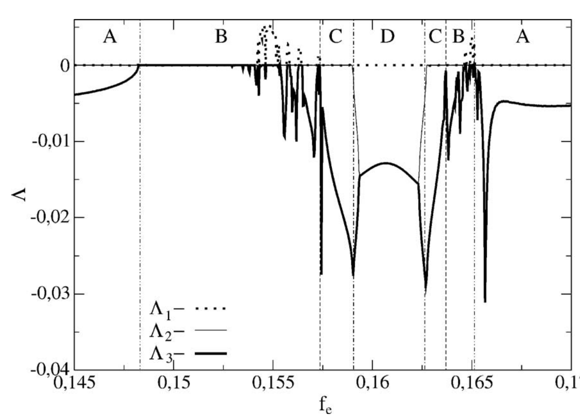

FIG. 5. Three largest Lyapunov exponents of system (3) as functions of the external force frequency f_e for $k_e=0.025$.

in the system. In the phase space, a two-dimensional torus corresponds to this regime for which two Lyapunov exponents are equal to zero (Fig. 5). The oscillators satisfy the resonant condition in region A.

The regime of mutual synchronization in region B (Figs. 1–3) is destroyed. The frequencies f_1 and f_2 are different, which is illustrated in Fig. 4(c). As a result, the motions with three independent frequencies f_1, f_2, and f_e are observed in region B. A three-dimensional torus corresponds to that regime for which three Lyapunov exponents are equal to zero (Fig. 5). The basic frequency f_e is varied, different partial resonances in the form of T^2 and even chaotic regimes can emerge on the three-dimensional torus. However, studying these bifurcations is not the aim of the present paper.

When reaching the region C, the following phenomenon takes place. The basic frequency of the first oscillator becomes locked by the external force. For that regime $f_e=f_1$, but $f_1 \neq f_2$. A resonant structure in the form of a two-dimensional torus appears on the three-dimensional torus, and two Lyapunov exponents are equal to zero. Our computations have shown that in this case the Poincaré section looks like a one-dimensional closed curve.

Finally, the regime of complete synchronization is realized in region D. The external force locks both frequencies of the interacting oscillators and the condition $f_e=f_1=f_2$ is satisfied. Only one Lyapunov exponent equals zero in region D, and the phase portrait shows an attractor in the form of a limit cycle.

The results presented above demonstrate that there is a great difference between the synchronization of a resonant limit cycle and the classical synchronization of periodic motions. In the resonant case, the external force first destroys the regime of initial mutual synchronization, then one of the basic frequencies becomes locked, and then the other one. As a result, the complete synchronization takes place and corresponds to the effect of winding number locking [Fig. 4(c), region D].

IV. BIFURCATIONS OF QUASIPERIODIC REGIMES IN NONAUTONOMOUS SYSTEM

Let now study in more detail the transition mechanisms of the oscillatory regimes in system (3) when the external force frequency is varied. We have calculated the bifurcation diagram on the parameters plane "amplitude versus frequency" of the external force (Fig. 6). Figures 2 and 3 correspond to the line $k_e=0.025$ in Fig. 6. Bifurcational lines l_{T^3} correspond to the transitions from region A to region B, lines l_p—from B to C, lines l_f—from C to D. Let us consider in more detail the bifurcational phenomena taking place on the lines l_{T^3}, l_p, and l_f.

The basic oscillatory regime realized in system (3) characterized by three independent frequencies $f_e \neq f_1 \neq f_2$ and is observed in region B. The corresponding attractor is a three-dimensional torus T^3. All the main bifurcations leading to initial resonant cycle synchronization are connected with the bifurcations of regime T^3.

Now we consider region B in which the stable three-dimensional torus T^3 exists. The bifurcations of the torus T^3 can be analyzed by a double Poincaré section.[8] In the classical Poincaré section the torus T^3 has the form of a two-dimensional torus T_{T^3}. In the double Poincaré section it corresponds to a closed invariant curve in the form of cycle L_{T^3}. A fixed point on that invariant curve corresponds to a resonant two-dimensional torus lying on the three-dimensional torus T^3.

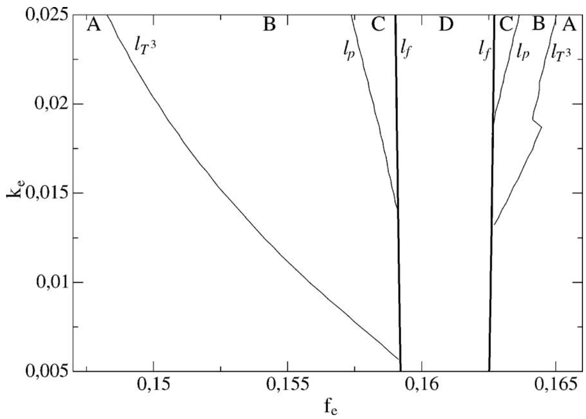

FIG. 6. Bifurcation diagram of system (3) on the parameter plane (f_e, k_e) for $m=0.1$, $\alpha_1=1$, $\alpha_2=1.0015$, $k=0.02$.

Let us consider the transition from region B into region A when crossing the line l_{T^3} in the bifurcation diagram (Fig. 6). The computation results in the double Poincaré section are shown in Fig. 7. The curve L_{T^3} corresponds to the regime of T^3 existing in region B. When reaching the bifurcation point (crossing the line l_{T^3} from region B into region A) on the curve L_{T^3}, a fixed point "saddle node" appears. In region A, that point breaks into a stable node and a saddle. In the double Poincaré section the classical saddle-node bifurcation takes place.

In the phase space of system (3), the phenomenon shown in Fig. 7 characterizes the appearance (disappearance) of a pair of two-dimensional tori on the three-dimensional torus T^3. One of them is stable (T_n^2) and the other one is saddle (T_{ns}^2).

Now we analyze the bifurcation transition from region B to region C when crossing the bifurcational lines l_p in Fig. 6.

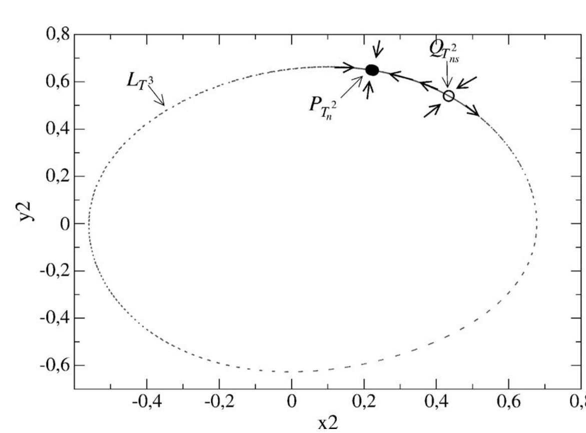

FIG. 7. Saddle-node bifurcation in the double Poincaré section when crossing the line l_{T^3} from region B into region A. L_{T^3} is an invariant curve, $P_{T_n^2}$ is a stable node, and $Q_{T_{ns}^2}$ is a saddle point. Parameters values: $f_e=1.482$, $k_e=0.025$, $m=0.1$, $\alpha_1=1$, $\alpha_2=1.0015$, and $k=0.02$.

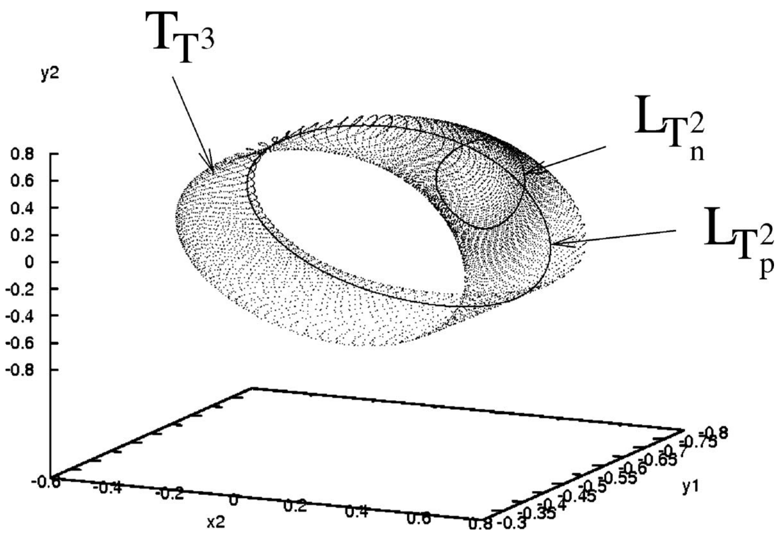

FIG. 8. Projections of Poincaré sections of the three-dimensional torus T_{T^3} (f_e=0.15) and resonant two-dimensional tori $L_{T_n^2}$ (f_e=0.1482) and $L_{T_p^2}$ (f_e=0.158).

Our computations have shown that on the line l_p the same saddle-node bifurcation takes place and in region C stable and saddle resonant two-dimensional tori appear on the surface of three-dimensional torus T^3. The calculation results are shown in Fig. 8. The Poincaré section of the three-dimensional torus T^3 is labeled as T_{T^3}. The closed invariant curve $L_{T_p^2}$ in Fig. 8 corresponds to the resonant stable two-dimensional torus existing in region C. The Poincaré section of the stable two-dimensional torus $L_{T_n^2}$ existing in region A is drawn for comparison. Saddle tori are not shown in Fig. 8. The tori T_n^2 and T_p^2 are different as they correspond to the different cases of partial synchronization. In region A, the frequencies f_1=f_2 and $f_e \neq f_1$, and in the region C, f_e=f_1 and $f_1 \neq f_2$.

Finally, we consider the bifurcation transition from region C to region D when crossing the lines l_f. That effect is followed by the second frequency locking f_2=f_e, and results in the regime of complete synchronization; i.e., f_1=f_2=f_e. The line l_f corresponds to the classical saddle-node bifurcation of resonant cycles lying on a two-dimensional torus. In the bifurcation point (on the line l_f) stable and saddle cycles appear on the two-dimensional torus T_p^2 being a resonant structure on T^3 and observed in region C. In region D on the line l_f, the system demonstrates stable periodic motions corresponding to the regime of complete synchronization. Phase projections of the two-dimensional torus T_p^2 (in region C) and the stable resonant cycle L_f on it (region D) are shown in Fig. 9.

V. SYNCHRONIZATION EFFECT ANALYSIS FOR VARIOUS VALUES OF WINDING NUMBER Θ AND COUPLING STRENGTH *k*

Our studies of synchronization effects have shown that the discussed case of resonance Θ=1:1 is the most general and rather complicated one from the viewpoint of the theory of bifurcation. It is also interesting to analyze synchronization effects for other values of the winding number corresponding to resonances Θ=m:n, where m, n=1,2,…. The effects of synchronization in system (3) must also depend on the coupling strength k between the oscillators. Such a dependence is quite important for understanding the mechanisms of synchronization of quasiperiodic oscillators in which the coupling parameter cannot be included explicitly or cannot be an independent parameter. In this connection we study the peculiarities of system (3), especially the bifurcation properties for various values of parameter k.

We consider the regime of the resonant limit cycle in system (2) with the winding number Θ=1:3 and synchronize it with the external periodic signal (3).

In the autonomous system (2), the regime of mutual synchronization is realized for the parameter values α_1=1, α_2=0.328, m=0.1, and k=0.005. A stable resonant cycle on a two-dimensional torus corresponds to that regime. Projection of the phase portrait and the power spectrum of that cycle are shown in Fig. 10. Now we add the external force to the system and vary f_e in the vicinity of f_2.

The results of our computations are shown in Fig. 11. The main difference between the resonance 1:1 and the shown case is that the complete synchronization is not observed. The second frequency is locked (f_e=f_2), while the frequency f_1 does not depend on the external force. This

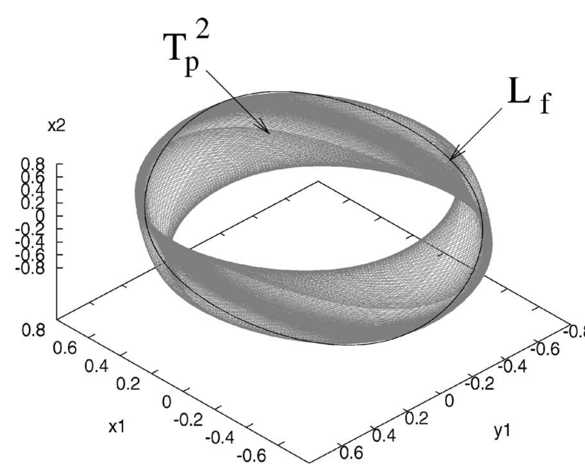

FIG. 9. Projections of phase portraits of the two-dimensional torus T_p^2 (gray) and the resonant limit cycle L_f (black) on it for the frequency values: f_e=0.1587 (T_p^2) and f_e=0.1592 (L_f).

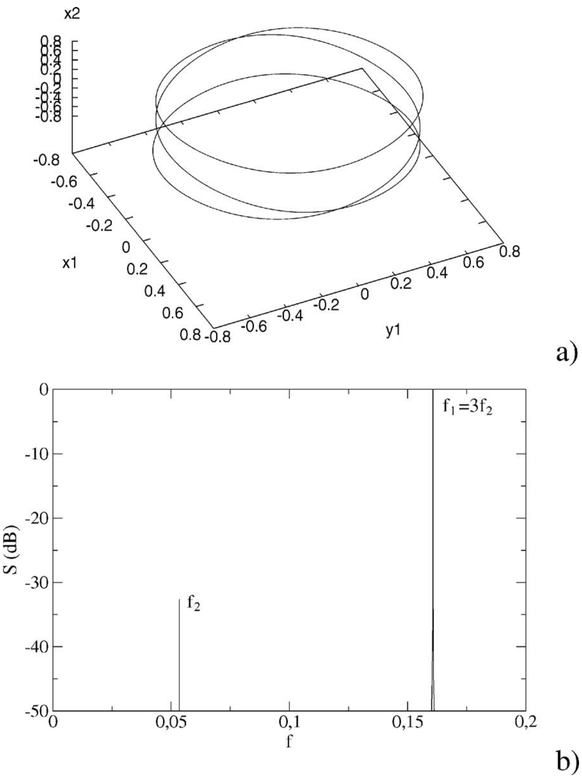

FIG. 10. The limit cycle in system (2) for the resonant condition $\Theta=1:3$ (a) and the corresponding power spectrum (b) obtained for the parameters values $\alpha_1=1$, $\alpha_2=0.328$, $m=0.1$, and $k=0.005$.

result is qualitatively the same as presented in Ref. 7. Similarly to the resonance 1:1 and if the frequency f_e is far from f_2, there exists region A in which a resonant two-dimensional torus $T^2_{n1:3}$ is realized on the surface of the three-dimensional

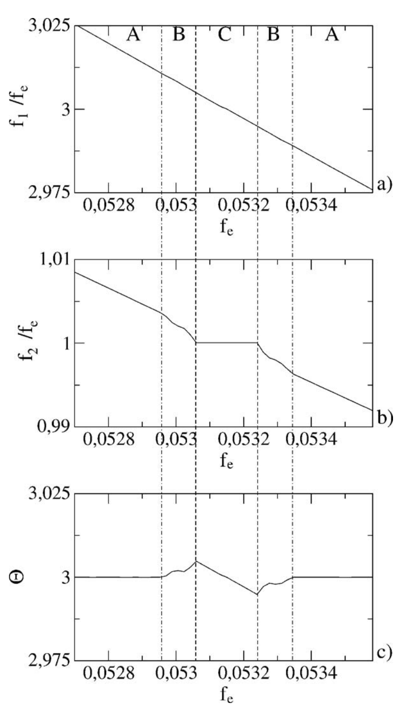

FIG. 11. Frequency relations f_1/f_e (a), f_2/f_e (b) and winding number $\Theta=f_1:f_2$ (c) as functions of the external force frequency f_e for $k_e=0.005$ and resonance 1:3.

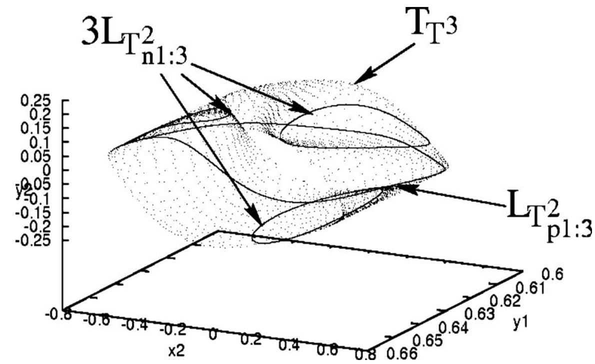

FIG. 12. Projections of Poincaré sections of two-dimensional tori in the form of cycles lying on the surface of the three-dimensional torus T_{T^3}: $3L_{T^2_{n1:3}}$ exists in region A (Fig. 11), $f_e=0.0527$, and $L_{T^2_{p1:3}}$ exists in region C (Fig. 11), $f_e=0.0531$

torus T^3. When the resonance 1:3 is destroyed, there appears the torus T^3 that exists in region B. The transition from region B to region C results in the appearance of a resonant two-dimensional torus $T^2_{p1:3}$. The latter corresponds to the regime of partial synchronization: $f_e=f_2$, $f_1\neq 3f_2$. The region D is not observed in the present case. The complete synchronization of the resonant limit cycle can take place by applying an additional external periodic signal with a frequency close to f_1. Projections of Poincaré sections of resonant two-dimensional tori corresponding to the regions A ($T^2_{n1:3}$) and C ($T^2_{p1:3}$) (Fig. 11) in the form of cycles L are shown in Fig. 12. These tori lie on the surface of the three-dimensional torus T^3 labeled in Fig. 12 as T_{T^3} in the Poincaré section.

The results shown in Figs. 11 and 12 confirm the data presented in Ref. 7 and thus give evidence of their generality.

The dependencies presented in Fig. 11 correspond to weak internal coupling $k=0.005$ in system (2). Let us clarify how the increase of internal coupling strength affects the synchronization phenomena in the system. The results of our computations are shown in Fig. 13 for $k=0.02$. When the internal coupling strength increases, one can observe region D being the region of complete synchronization of the resonant cycle on the torus. Our investigations have shown that when $k>0.02$, region D becomes wider and the system's behavior is qualitatively similar to the case of the resonance 1:1 (Fig. 4).

VI. CONCLUSIONS

Our results of numerical modeling of limit cycle synchronization corresponding to the resonance on a two-dimensional torus demonstrate the following:

(1) A limit cycle on a two-dimensional torus cannot be generally synchronized by an external harmonic force. The influence of the external force results in the resonance destruction and in the transition to quasiperiodic motions with three independent frequencies (regions B in Figs. 4, 6, 11, and 13). Each of the basic frequencies of the system (2) then becomes successively locked.
(2) For resonances $\Theta=p:q$ ($p=1$, $q\geq 3$) and weak coupling between the oscillators, the external harmonic force syn-

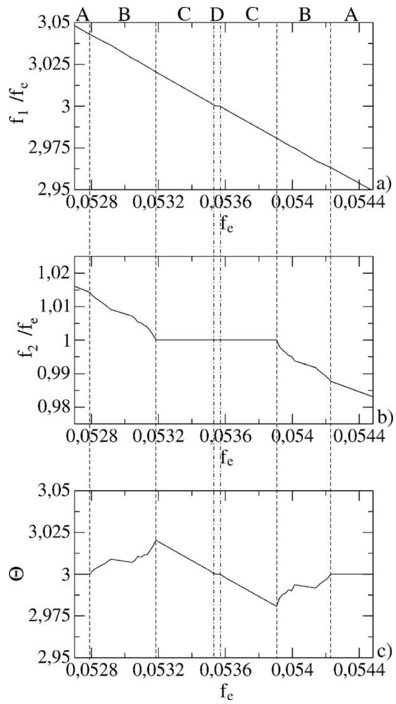

FIG. 13. Frequency relations f_1/f_e (a), f_2/f_e (b), and winding number $\Theta = f_1 : f_2$ (c) as functions of the external force frequency f_e for $k_e = 0.005$, $k = 0.02$ and resonance 1:3.

chronizes only one of the basic frequencies of the system (2); i.e., f_1 or f_2. However, when the internal coupling strength increases, both basic frequencies can be successively locked (Fig. 13).

(3) The resonant limit cycle on the two-dimensional torus can always be synchronized by the external two-frequency quasiperiodic force with a close winding number value. In that case the effect of winding number locking established in Ref. 6 can take place.

One of the basic frequencies of the two-frequency quasiperiodic motions is synchronized as a result of the saddle-node bifurcation of stable and saddle two-dimensional tori on a three-dimensional torus. That bifurcation is detected for the first time and is of scientific interest for the qualitative theory of differential equations.

ACKNOWLEDGMENTS

The authors thank Professor T. Vadivasova for useful discussions and constructive remarks. This work was partially supported by the Russian Ministry of Education and Science in the framework of the program "Development of Scientific Potential of the Higher School." J.K. thanks the SFB 555 DFG for financial support.

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