Peculiarities of synchronization of a resonant limit cycle on a two-dimensional torus

V. Anishchenko* and S. Nikolaev

Institute of Nonlinear Dynamics, Department of Physics,

Saratov State University, 410012 Saratov, Russia

J. Kurths

Group of Nonlinear Dynamics, Institute of Physics,

University of Potsdam, D-14415, Potsdam, Germany

Abstract

The peculiarities of external synchronization of a resonant limit cycle on a torus are studied in an autonomous oscillator of quasiperiodic oscillations with two basic frequencies. We show numerically and experimentally that in the resonance conditions the synchronization effect takes place only at one of the two basic frequencies of the system, while the oscillations at the second basic frequency remain unsynchronized. Our results convincingly indicate a principal difference between synchronization of the resonant limit cycle on the torus and of a typical limit cycle. This is in contrast to the well-established theory of synchronization of a limit cycle. This finding opens new strategies for controlling systems with multiple time scales.

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*Electronic address: `wadim@chaos.ssu.runnet.ru`

Quasiperiodic oscillations with two and more basic frequencies are widely encountered in the contemporary natural sciences. They appear when electro-magnetic oscillations are modulated by information signals in radio-engineering, accompany the transition to turbulence in fluid flows (hydrodynamics), describe the motion of planets (celestial mechanics), etc. Quasiperiodic oscillations describe biophysical, ecological and even social evolutionary processes such as the cardio-respiratory system, photosynthesis: day-night cycle and Calvin cycle and others. In phase space they are associated with limit sets or attractors in the form of n-dimensional tori. The analysis of stability, bifurcations and synchronization of quasiperiodic oscillations are determined by resonances and bifurcations of n-dimensional tori. This is a rather complex and in many ways unsolved problem. In our research we deal with the case of quasiperiodic oscillations whose image in the phase space represents an attracting limit set in the form of a two-dimensional ($2D$) torus.

In our case we study quasiperiodic oscillations in an autonomous dissipative dynamical system that realizes the regime of stable self-sustained oscillations with two basic frequencies. It is important to note that non-autonomous two-frequency oscillations that are observed when a limit cycle system is periodically driven cannot be used here because in such systems only one of two frequencies (the oscillation frequency on a limit cycle) depends on parameters of a system. The external forcing frequency is not defined by the system properties.

An autonomous dissipative dynamical system in R^4 that can demonstrate stable two-frequency oscillations can be described by the following system of equations [1, 2]:

$$\begin{aligned}
\dot{x} &= mx + y - x\varphi - dx^3, \\
\dot{y} &= -x, \\
\dot{z} &= \varphi, \\
\dot{\varphi} &= -\gamma\varphi + \gamma\Phi(x) - gz.
\end{aligned} \tag{1}$$

where m, d, γ, and g are system parameters, and $\Phi(x)$ is a nonlinear function that is defined as follows:

$$\Phi(x) = H(x)x^2, \tag{2}$$

where $H(x)$ is the Heaviside function. The oscillator of two-frequency oscillations (1) is a suitable model to study the regularities of external and mutual synchronization of quasiperiodic oscillations and to establish a new effect – the winding number locking on a $2D$ torus [2]. The results presented in [2] were obtained for regimes of non-resonant two-frequency oscillations satisfying the inequality:

$$\frac{f_1}{f_2} \neq \frac{p}{q}, \tag{3}$$

where $p, q = 1, 2, \ldots, k$, f_1 and f_2 are the basic oscillation frequencies of the oscillator. The resonant case on a $2D$ torus when $f_1/f_2 = p/q$ was excluded in [2], because we expected that the resonance on a $2D$ torus corresponds to a stable limit cycle and its synchronization conditions are well studied and described in detail in books (see, for example, [3–5]). However, we show in this letter, this is not true. We especially demonstrate that a limit cycle not lying on a torus and a resonant limit cycle on the $2D$ torus respond to an external periodic force in a completely different way. We present numerical and experimental results that describe these peculiarities of the resonant limit cycle synchronization on a $2D$ torus.

Turn to the system (1). We set the system parameters, $m = 0.096$, $g = 0.257$, $\gamma = 0.2$, and $d = 0.001$ and choose initial conditions in the vicinity of the coordinate origin. In this case the system (1) has a stable quasiperiodic solution with two irrationally related frequencies f_1 and f_0. A projection of the corresponding ergodic $2D$ torus is presented in Fig. 1,a and the power spectrum of $x(t)$ in Fig. 1,b. If we change the parameter g that controls the winding number, then for $g = 0.263$ a resonant mode is observed on the torus resulting in $f_1 : f_0 = 1 : 4$ (Fig. 1,c,d). In this regime we have a stable limit cycle on the torus (Fig. 1,c) and its power spectrum (Fig. 1,d) contains only one basic frequency f_1 and its harmonics $2f_1$, $3f_1$, $4f_1 = f_0$ etc. If the way of limit cycle emerge is unclear it is impossible to claim that the limit cycle realized by any dynamical system lies on the $2D$ torus surface. We deal with a typical stable periodic oscillatory regime with period $T_0 = 1/f_1$ and its power spectrum contains frequency f_1 and its harmonics nf_1. Let us attempt to synchronize this cycle by an external harmonic signal with frequency $f_{\text{ex}} = f_1 + \Delta$, where Δ is a small frequency mismatch.

Now we study the influence of the additive external force $k \sin(2\pi f_{\text{ex}})$ on system (1):

$$\begin{aligned}
\dot{x} &= mx + y - x\varphi - dx^3 + k\sin(2\pi f_{ex}), \\
\dot{y} &= -x, \\
\dot{z} &= \varphi, \\
\dot{\varphi} &= -\gamma\varphi + \gamma\Phi(x) - gz.
\end{aligned} \tag{4}$$

Which is for $k = 0$ in the resonant regime. We analyze a weak external forcing ($k = 0.01$). We calculate the power spectrum of $x(t)$ from eq. (4) as the external signal frequency f_{ex}

is varied. The numerical results are pictured in Fig. 2. We find that $f_{\rm ex}$ locks the internal frequency f_1 in the region $f_{\rm ex} \simeq 0.0381 \div 0.0385$. In the synchronization region (Fig. 2,a) the modulation frequency f_1 is locked by the external force and the condition $f_1/f_{\rm ex} = 1$ is fulfilled. It is important to emphasize that at the same time the frequency f_0 is not synchronized by the external force Fig. 2,b); f_0 does not essentially change both outside the synchronization region of the frequency f_1 and inside it at all. In other words, the frequency f_0 does not respond to the change of the external force frequency $f_{\rm ex}$. If we would deal with a typical limit cycle, then the spectral line at the frequency $f_0 = 4f_1$, as well as at any harmonic nf_1, would demonstrate the synchronization effect. However, such a situation is not realized in system (1). In the autonomous quasiperiodic self-sustained oscillator the two frequencies f_1 and f_0 correspond to the different oscillatory modes, even being rationally related $(1:4)$. They remain independent in a sense that an external force effects them in a different way. Next this fact will be also shown for second harmonic. Let $f_{\rm ex} = 2f_1 + \Delta$. Our findings practically repeat the results presented in Fig. 2. The frequency f_1 is locked by the double frequency signal within a finite region $(0.0763 \div 0.0766)$. With this, the frequency f_0 remains unsynchronized as in case of Fig. 2.

Next, we confirm this phenomenon experimentally. We use an electronic generator of quasiperiodic motions [2]. It can be modeled by (1). We have chosen the regime of oscillations which corersponds to the resonance $1:3$. We synchronize a resonant $1:3$ limit cycle (Fig. 3,a) in order to demonstrate that the results shown in Fig, 2 does not depend on the winding number. If we apply the periodic force to the electronic oscillator in the regime of periodic motions (in our case it is resonant limit cycle (Fig. 3,a)) then outside the synchronization region ($f_{\rm ex} = 2.9$ kHz) a $2D$ torus is observed. The projection of it is shown on Fig. 3,b). Fig. 3,c illustrates a projection of the torus existing inside the synchronization region of frequency f_1 ($f_{\rm ex} = 3.4$ kHz). The measurement results being similar to the numerical findings shown in Fig. 2 are presented in Fig. 3,d,e. The effect of external synchronization of frequency f_1 is illustrated by Fig. 3,d, and Fig. 3,e confirms that the oscillator frequency f_0 does not depend on the external signal frequency $f_{\rm ex}$.

The physical and numerical experiments in current paper consider the case of external force frequency $f_{\rm ex}$ close to modulation frequency f_1. Generally the same results can be obtained for the basic frequency f_0 if the external frequency $f_{\rm ex}$ is close to it.

Now, we study the mechanism behind the phenomena shown in Figs. 2 and 3. We analyze how the limit sets of system (4) evolve as the external frequency $f_{\rm ex}$ changes by considering their Poincaré sections, for which secant surface satisfies the condition $x(t) = 0$. In the unforced system the initial torus (Fig. 4,a) looks like a closed invariant curve l_1 (Fig. 4,a) in the Poincaré section and the resonant case $1:4$ ("$\bullet$", Fig. 4,a,b) is identified by the appearance of four stable fixed points that correspond to the Poincaré section of the resonant cycle (Fig. 1,c). When the external force starts acting with frequency $f_{\rm ex} \neq f_1$, a new $2D$ torus is born in the vicinity of the resonant cycle (Fig. 3,b). In the Poincaré section this torus is represented by four invariant curves l_2 in the vicinity of the corresponding fixed points (Fig. 4,b). All these transformations occur in complete agreement with the classical mechanism taking place for the limit cycle in the Van der Pol oscillator when approaching the synchronization region [6]. However, the classical scenario is further violated. According to the classical theory, when entering the region of frequency f_1 locking (Fig. 2) the resonance on the torus must correspond to the emergence of a fixed point on the invariant curves l_2. However, this is not observed in our case. Moreover, the invariant curves l_2 undergo some complex rebuildings that result in the appearance of an invariant curve l_3 in the synchronization region of frequency f_1 in the Poincaré section (Fig. 4,c). This curve corresponds to the torus resembling the initial one (Fig. 4,a). We can at least infer that the invariant curve l_3 is not topologically related with the curves l_2. We have also calculated the full spectrum of Lyapunov exponents for all the cases presented in Fig. 4 (naturally excluding the resonant cycle). Our calculations have shown that the Lyapunov spectrum contains two zero maximum exponents.

From a physical viewpoint, the obtained results can be explained as follows. The quasiperiodic oscillator (1) in fact represents an autonomous system of two interacting nonlinear oscillators, as it was shown in [2]. The internal coupling of oscillators ensures the generation of two-frequency oscillations. Their properties depend on the controlling parameters. For certain parameter values, the frequencies of oscillators can be mutually synchronized, i.e., the resonance $f_1 : f_0 = p : q$ takes place. Physically, the basic frequencies f_1 and f_0, even being rationally connected, correspond to a different oscillatory modes. In the presence of external periodic force each of the oscillators can be synchronized independently. Our experiments have shown that the resonant limit cycle can be synchronized on the torus when the system is driven by an external two-frequency signal including $f_1' = f_1 + \Delta$ and $f_2' = f_0 + \Delta$. In this case both oscillators will be synchronized in the resonant mode

$p : q$ and the winding number will be locked. The latter phenomenon was established and described in [2].

The bifurcational mechanism of resonant cycle synchronization on a $2D$ torus described in this Letter is a rather complicated problem of the qualitative theory. But we hope to study it in future.

New phenomenon of synchronization of quasiperiodic dynamics can be used to diagnose the presence of a resonant torus in a system. If the torus exists, then its basic frequencies will demonstrate the effect of synchronization independently (Figs. 2 and 3). In case when a system demonstrates a complex multi-fold (loop) limit cycle not lying on a torus, the synchronization at frequency f_1 and at any of its harmonics nf_1 can lead to the effect when all harmonics in the spectrum are locked. We have observed this effect experimentally.

The obtained result can be applied to the dynamics of multi-scale systems control.

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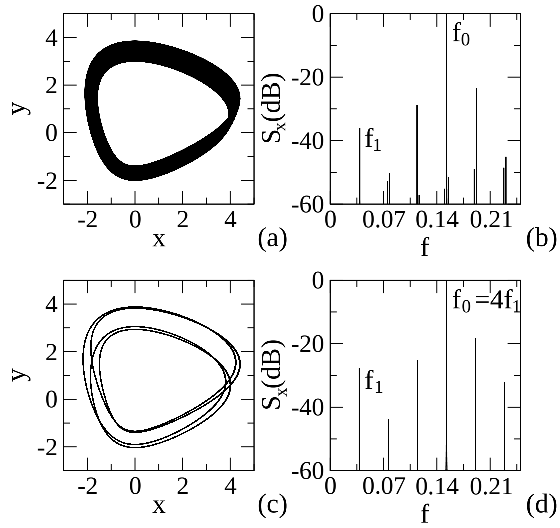

FIG. 1: Non-resonant (a,b) and resonant (c,d) quasiperiodic oscillations with two frequencies f_1 (modulation frequency) and f_0 (basic frequency). (a) Projection of a non-resonant torus on the plane (x, y); (b) power spectrum of $x(t)$ oscillations for the non-resonant case (a); (c) limit cycle on a torus in the resonant case $f_1 : f_0 = 1 : 4$, and (d) power spectrum of the resonant cycle

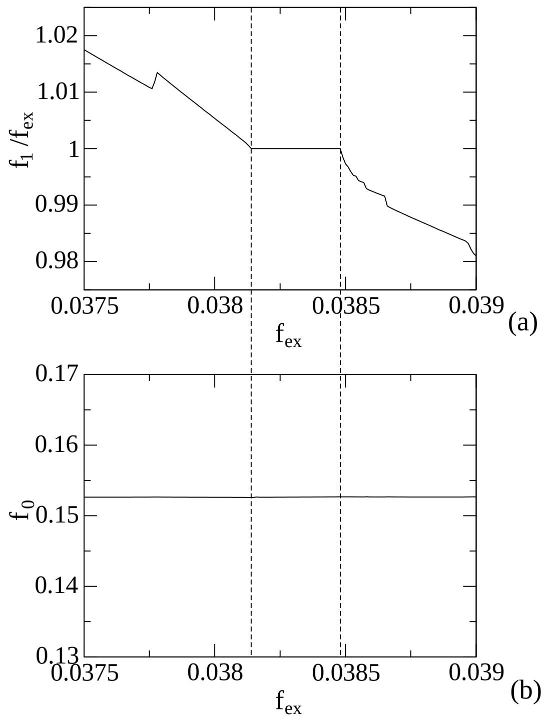

FIG. 2: Calculation results of the frequency relation f_1/f_{ex} (a) and of the frequency f_0 (b) as a function of the external signal frequency in system (4) for $m = 0.096$, $g = 0.263$, $\gamma = 0.2$, $d = 0.001$, and $k = 0.01$

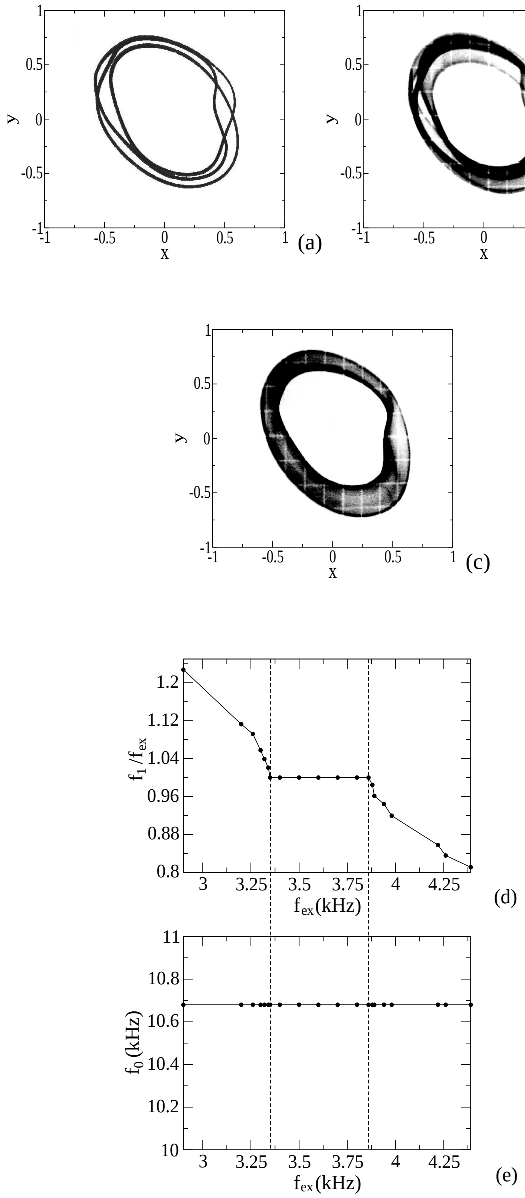

FIG. 3: Experimental results. (a) Phase portrait projection of an autonomous limit cycle on a torus in the resonant case 1 : 3, (b) $2D$ torus resulted from the periodically driven limit cycle (outside the synchronization region, $f_{\rm ex} = 2.9$ kHz), c) $2D$ torus projection inside the synchronization region of frequency f_1 ($f_{\rm ex} = 3.4$ kHz). Frequency relation $f_1/f_{\rm ex}$ (d) and f_0 (e) as a function of the external force frequency $f_{\rm ex}$ (lines with black points were determined experimetally).

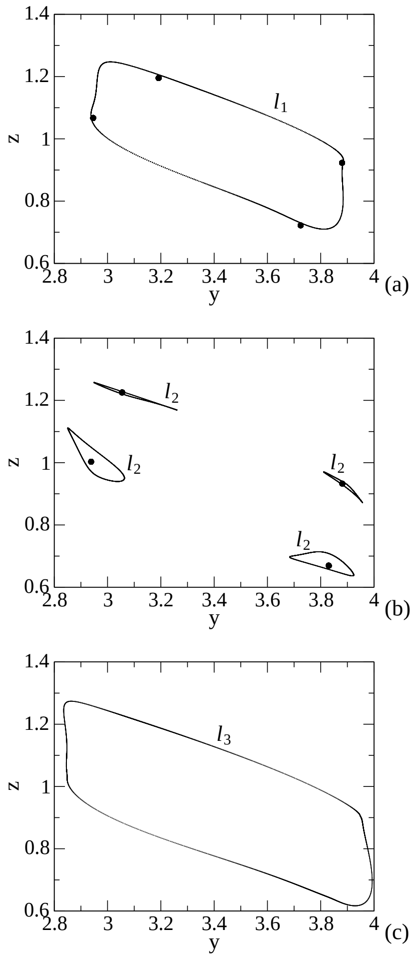

FIG. 4: Projections of the Poincaré sections. (a) Limit cycle on a torus generated by the system without external driving in the resonant case 1 : 4 ("•", $g = 0.261$) and torus generated in the regime of quasiperiodic oscillations (l_1, $g = 0.262$); (b) torus being subject to the external force at the frequency outside the synchronization region (l_2, $g = 0.262$, $f_{\text{ex}} = 0.03758$) and resonant torus generated by system without external force ("•", $g = 0.263$); (c) torus in the synchronization regime (l_3, $g = 0.263$, $f_{\text{ex}} = 0.0383$)