Universality in the Synchronization of Weighted Random Networks

Changsong Zhou,[1] Adilson E. Motter,[2,3] and Jürgen Kurths[1]

[1]*Institute of Physics, University of Potsdam PF 601553, 14415 Potsdam, Germany*

[2]*CNLS and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

[3]*Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208, USA*

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Realistic networks display not only a complex topological structure, but also a heterogeneous distribution of weights in the connection strengths. Here we study synchronization in weighted complex networks and show that the synchronizability of random networks with a large minimum degree is determined by two leading parameters: the mean degree and the heterogeneity of the distribution of node's intensity, where the intensity of a node, defined as the total strength of input connections, is a natural combination of topology and weights. Our results provide a possibility for the control of synchronization in complex networks by the manipulation of a few parameters.

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Over the past few years, the analysis of complex systems from the viewpoint of networks has become an important interdisciplinary issue [1]. It has been shown that physical and dynamical processes, such as cascading failures [2], epidemic spreadings [3], and network synchronization [4–9], are strongly influenced by the structure of the underlying network. Previous work on synchronization has focused mainly on the influence of the topology of the connections by assuming that the coupling strength is uniform. However, synchronization is influenced not only by the topology, but also by the strength of the connections [9]. Most complex networks where synchronization is relevant are indeed weighted. Examples include brain networks [10], networks of coupled populations in the synchronization of epidemic outbreaks [11], and technological networks whose functioning relies on the synchronization of interacting units [12]. The distribution of connection weights in real networks is often highly heterogeneous [13]. The study of synchronization in weighted networks is thus of substantial interest.

In this Letter, we address this question in random networks with weighted coupling schemes motivated by real networks. Our main result is the uncovering of a universal formula that describes with good approximation the synchronizability of identical oscillators solely in terms of the mean degree and the heterogeneity of the node's intensity, irrespective of the degree distribution and other topological properties. The intensity of a node, defined as the sum of the strengths of all input connections of that node, incorporates both topological and weighted properties and raises as a very important parameter controlling the synchronizability. In particular, it follows that the synchronizability is significantly enhanced when the heterogeneity of the node's intensities is reduced.

The dynamics of a general weighted network of N coupled identical oscillators is described by

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} W_{ij}A_{ij}[\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)], \quad (1)$$

$$= \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} G_{ij}\mathbf{H}(\mathbf{x}_j), \qquad i = 1, \ldots, N, \quad (2)$$

where $\mathbf{F} = \mathbf{F}(\mathbf{x})$ governs the dynamics of each individual oscillator, $\mathbf{H} = \mathbf{H}(\mathbf{x})$ is the output function, and σ is the overall coupling strength. Here $G = (G_{ij})$ is the coupling matrix combining both topology [adjacency matrix $A = (A_{ij})$] and weights [weight matrix $W = (W_{ij})$, $W_{ij} \geq 0$]: $G_{ij} = \delta_{ij}S_i - W_{ij}A_{ij}$, where $S_i = \sum_{j=1}^{N} W_{ij}A_{ij}$ denotes the intensity of node i. The rows of G have zero sum, and this ensures that the completely synchronized state $\{\mathbf{x}_i = \mathbf{s}, \forall i | \dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})\}$ is an invariant manifold of Eq. (2). In this work, we focus on the class of weighted networks where G is diagonalizable and has real eigenvalues. As it will be shown shortly, in this case the synchronizability of the networks can be characterized by the properties of the eigenvalues, without referring to specific forms of $\mathbf{F}$ and $\mathbf{H}$. This applies, in particular, to the important class of networks where G can be written as $G = \mathcal{B}C$ for $\mathcal{B}$ a nonsingular diagonal matrix and C a symmetric, zero row-sum matrix. We assume that matrix A is binary and symmetric, and that the connection weights and asymmetries are incorporated into W.

The linear stability of the synchronized states can be studied by diagonalizing the variational equations of Eq. (2) into N blocks of the form [14]

$$\dot{\xi}_l = [D\mathbf{F}(\mathbf{s}) - \sigma\lambda_l D\mathbf{H}(\mathbf{s})]\xi_l, \qquad l = 1, \ldots, N, \quad (3)$$

which are different only by λ_l, the lth eigenvalue of G, ordered as $0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N$. The eigenvalues of G are nonnegative because $G_{ii} = -\sum_{j\neq i} G_{ij} \geq 0$ and $\lambda_1 = 0$ because $\sum_j G_{ij} = 0$ for all i. The stability of $\mathbf{s}$ is determined by λ_l and the master stability function [14], i.e., the largest Lyapunov exponent Λ of the generic variational equation $\dot{\xi} = [D\mathbf{F}(\mathbf{s}) - \epsilon D\mathbf{H}(\mathbf{s})]\xi$. For many oscillatory dynamical systems [6,14], Λ is negative in a single, finite interval $\epsilon_1 < \epsilon < \epsilon_2$, where the thresholds ϵ_1 and ϵ_2 are determined only by $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{s}$. The network is thus synchronizable for some σ iff the condition $\epsilon_1 < \sigma\lambda_l < \epsilon_2$ is satisfied so that $\Lambda(\sigma\lambda_l) < 0$ for all $l \geq 2$. This is equivalent to the condition

$$R \equiv \lambda_N/\lambda_2 < \epsilon_2/\epsilon_1, \tag{4}$$

where the eigenratio R depends only on the network structure (G), and ϵ_2/ϵ_1 depends only on the dynamics. From these, it follows that the smaller the eigenratio R, the more synchronizable the network and vice versa [6]. Another measure of synchronizability is the cost C involved in the couplings of the network [9]. When Eq. (4) is satisfied, the synchronized state is linearly stable for $\sigma > \sigma_{\min} \equiv \epsilon_1/\lambda_2$. The cost C is the total input strength of the connections of all nodes at the synchronization threshold: $C = \sigma_{\min}\sum_{i,j} W_{ij}A_{ij} = \sigma_{\min}\sum_{i=1}^{N} S_i$. The normalized cost

$$C_0 \equiv C/(N\epsilon_1) = \Omega/\lambda_2, \tag{5}$$

where $\Omega = \sum_{i=1}^{N} S_i/N$, does not depend on the dynamics ($\mathbf{F}$, $\mathbf{H}$, and $\mathbf{s}$) and can be used as a complementary parameter of synchronizability [9]. Here we characterize the synchronizability of the networks using both the eigenratio R and the cost C_0.

Previous work has obtained bounds for the eigenvalues of *unweighted* networks ($W_{ij} = 1$) (see [15] for a review). Such bounds, however, are not tight and may provide limited information about the actual synchronizability of complex networks. Here we aim at obtaining a more quantitative approximation of the synchronizability for a class of weighted random networks with real spectra, which includes unweighted networks as a special case. Our analysis is based on the combination of a mean field approximation and new graph spectral results.

First, in random networks with $k_{\min} \gg 1$, close to a synchronized state Eq. (1) can be approximated as

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma(S_i/k_i)\sum_{j=1}^{N} A_{ij}[\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)]. \tag{6}$$

The reason is that each oscillator j receives signals from a large and random sample of other oscillators in the network and $\mathbf{x}_j$ is not affected directly by the individual output weights W_{ij}. Consequently, we may assume that W_{ij} and $\mathbf{H}(\mathbf{x}_j)$ are statistically uncorrelated and that $\sum_{j=1}^{N} W_{ij}A_{ij}\mathbf{H}(\mathbf{x}_j) \approx (1/k_i)\sum_{j=1}^{N} W_{ij}A_{ij}\sum_{j=1}^{N} A_{ij}\mathbf{H}(\mathbf{x}_j) = S_i\bar{\mathbf{H}}_i$ if $k_i \gg 1$ [16]. Here $\bar{\mathbf{H}}_i = (1/k_i)\sum_{j=1}^{N} A_{ij}\mathbf{H}(\mathbf{x}_j)$ is the local mean field.

Now, if the network is sufficiently random, the local mean field $\bar{\mathbf{H}}_i$ can be approximated by the global mean field of the network, $\bar{\mathbf{H}}_i \approx \bar{\mathbf{H}} = (1/N)\sum_{j=1}^{N} \mathbf{H}(\mathbf{x}_j)$. Moreover, close to the synchronized state $\mathbf{s}$, we may assume $\bar{\mathbf{H}}_i \approx \mathbf{H}(\mathbf{s})$, and the system is approximated as

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma S_i[\mathbf{H}(\mathbf{s}) - \mathbf{H}(\mathbf{x}_i)], \qquad i = 1, \ldots, N, \tag{7}$$

indicating that the oscillators are decoupled and forced by a common oscillator $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$ with forcing strength proportional to the intensity S_i. The variational equations of Eq. (7) have the same form of Eq. (3), except that λ_l is replaced by S_l. If there exists some σ satisfying $\epsilon_1 < \sigma S_l < \epsilon_2$ for all l, then all the oscillators are synchronizable by the common driving $\mathbf{H}(\mathbf{s})$, corresponding to a complete synchronization of the whole network. These observations suggest that the eigenratio and the cost can be approximated as

$$R \approx S_{\max}/S_{\min}, \qquad C_0 \approx \Omega/S_{\min}, \tag{8}$$

where $S_{\min}$, $S_{\max}$, and Ω are the minimum, maximum, and mean intensities, respectively.

Next we present tight bounds for the above approximation. Equation (6) means that the coupling matrix G is replaced by the new matrix $G^a = (G^a_{ij})$, with $G^a_{ij} = \frac{S_i}{k_i} \times (\delta_{ij}k_i - A_{ij})$. G^a can be written as $G^a = S\hat{G} = SD^{-1}(D - A)$, where $S = (\delta_{ij}S_i)$ and $D = (\delta_{ij}k_i)$ are the diagonal matrices of intensities and degrees, respectively, and $\hat{G}$ is the normalized Laplacian matrix [17]. Importantly, now the contributions from the topology and weight structure are separated and accounted by $\hat{G}$ and S, respectively. We can show that the largest and smallest nonzero eigenvalues of matrix G^a are bounded by the eigenvalues μ_l of $\hat{G}$ as

$$S_{\min}\mu_2 c \leq \lambda_2 \leq S_{\min}c', \qquad S_{\max} \leq \lambda_N \leq S_{\max}\mu_N, \tag{9}$$

where c and c' can be approximated by 1 for most large complex networks of interest, such as realistic scale-free networks (SFNs). The proof is involved and long, and the details will be presented elsewhere. If the network is sufficiently random, the spectrum of $\hat{G}$ tends to the semicircle law for large networks with arbitrary expected degrees [17], provided that $k_{\min} \gg \sqrt{K}$, and $\max\{1 - \mu_2, \mu_N - 1\} = [1 + o(1)]\frac{2}{\sqrt{K}}$ for $k_{\min} \gg \sqrt{K}\ln^3 N$, where K is the mean degree. From these, it follows that

$$\mu_2 \approx 1 - 2/\sqrt{K}, \qquad \mu_N \approx 1 + 2/\sqrt{K}, \tag{10}$$

which we find to provide a good approximation under the weaker condition $k_{\min} \gg 1$, regardless of the degree distribution. From Eqs. (9) and (10), we have the following approximations for the bounds of R and C_0:

$$\frac{S_{\max}}{S_{\min}} \leq R \leq \frac{S_{\max}}{S_{\min}}\frac{1 + 2/\sqrt{K}}{1 - 2/\sqrt{K}}, \tag{11}$$

$$\frac{\Omega}{S_{\min}} \leq C_0 \leq \frac{\Omega}{S_{\min}}\frac{1}{1 - 2/\sqrt{K}}. \tag{12}$$

For the case of unweighted networks ($S_i = k_i$, $\Omega = K$), the bounds in Eq. (11) are much tighter than those reviewed in Ref. [15].

The bounds in Eqs. (11) and (12) show that the contribution of the network topology is mainly accounted by the mean degree K. Therefore, for a given K, the synchronizability of random networks with large $k_{\min}$ is expected to be well approximated by the following universal formula:

$$R = A_R \frac{S_{\max}}{S_{\min}}, \qquad C_0 = A_C \frac{\Omega}{S_{\min}}, \tag{13}$$

where the prefactors A_R and A_C are expected to be close to 1. In the case of matrix G^a with uniform intensity ($S_i = 1 \forall i$), they are given by the upper bounds, $A_R = \frac{1+2/\sqrt{K}}{1-2/\sqrt{K}}$ and $A_C = \frac{1}{1-2/\sqrt{K}}$, and $A_R \to 1$ and $A_C \to 1$ in the limit $K \to \infty$. Formula (13) is consistent with the approximation in Eq. (8) and indicates that the synchronizability of these networks is primarily determined by the heterogeneity of the intensities.

Our numerical results on various weighted and unweighted networks have confirmed this universal formula. First we consider the following weighted coupling scheme:

$$W_{ij} = S_i/k_i, \tag{14}$$

in which the intensities S_i follow an arbitrary distribution not necessarily correlated with the degrees. In this case, Eqs. (6) and (1) are identical and $G^a = G$. This weighted coupling scheme includes many previously studied systems as special cases. If $S_i = k_i \forall i$, it corresponds to the widely studied case of unweighed networks [5,6,8]. In the case of fully uniform intensity ($S_i = 1 \forall i$), it accommodates a number of previous studies about synchronization of coupled maps [4,7]. The weighted scheme studied in [9], $W_{ij} = k_i^{\theta}$, is another special case of Eq. (14) where $S_i = k_i^{1+\theta}$.

We have applied the weighted scheme to various network models: *(i) Growing SFNs with aging* [18]. Starting with $2m+1$ fully connected nodes, at each time step we connect a new node to m existing nodes according to the probability $\Pi_i \sim k_i \tau_i^{-\alpha}$, where τ_i is the age of the node. The minimum degree is then $k_{\min} = m$ and the mean degree is $K = 2m$. For the aging exponent $-\infty < \alpha \leq 0$, this growing rule generates SFNs with a power-law tail $P(k) \sim k^{-\gamma}$ and scaling exponent in the interval $2 < \gamma \leq 3$ [18], as in most real SFNs. For $\alpha = 0$, we recover the usual Barabási-Albert (BA) model [19], which has $\gamma = 3$. *(ii) Random SFNs* [20]. Each node is assigned to have a number $k_i \geq k_{\min}$ of "half-links" according to the distribution $P(k) \sim k^{-\gamma}$. The network is generated by randomly connecting these half-links to form links, prohibiting self- and repeated links. *(iii) K-regular random networks.* Each node is randomly connected to K other nodes.

We present results for two different distributions of intensity S_i which are uncorrelated with the distribution of degree k_i: (1) a uniform distribution in $[S_{\min}, S_{\max}]$; (2) a power-law distribution, $P(S) \sim S^{-\Gamma}$, $S \geq S_{\min}$, where $S_{\min}$ is a positive number. Consistently with the prediction of the universal formula, if $k_{\min} \gg 1$, the eigenratio R collapses into a single curve for a given K when plotted as a function of $S_{\max}/S_{\min}$ [Fig. 1(a)], irrespective of the distributions of k_i and S_i. The same happens for the cost C_0 as a function of $\Omega/S_{\min}$ [Fig. 1(b)]. The behavior of the fitting parameters A_R and A_C is shown in the insets of Fig. 1. For uniform intensity, they are very close to the upper bounds. They approach very quickly 1 when the intensities become more heterogeneous ($S_{\max}/S_{\min} > 3$). Therefore, Eq. (13) with $A_R = A_C = 1$ [Fig. 1, solid lines] provides a good approximation of the synchronizability for any large K if the intensities are not very homogeneous.

In more realistic networks, including scientific collaboration networks [13], metabolic networks [21], and airport networks [13,21], it has been shown that the weight W_{ij} of a connection between nodes i and j is strongly correlated with the product of the corresponding degrees as $\langle W_{ij} \rangle \sim (k_i k_j)^{\theta}$. Here θ depends on the specific network under study. Motivated by these observations, we analyze the weighted coupling [21]:

$$W_{ij} = (k_i k_j)^{\theta}, \tag{15}$$

where the weights are defined for the connections of a given network topology and θ is a tunable parameter. θ controls the heterogeneity of the intensity S_i and the correlation between S_i and k_i, since $S_i = k_i^{1+\theta} \langle k_j^{\theta} \rangle_i$, where $\langle k_j^{\theta} \rangle_i = (1/k_i) \sum k_j^{\theta}$ is approximately constant for $k_i \gg 1$ when the degree correlations can be neglected. Variations of θ have significant impact on the synchronizability of SFNs [Fig. 2]. However, as shown in the insets of Fig. 2 for various networks and θ values, R and C_0 collapse again to the universal curves when regarded as functions of $S_{\max}/S_{\min}$ and $\Omega/S_{\min}$, respectively.

We emphasize that these results also hold for unweighted random networks. As shown in Fig. 3 for un-

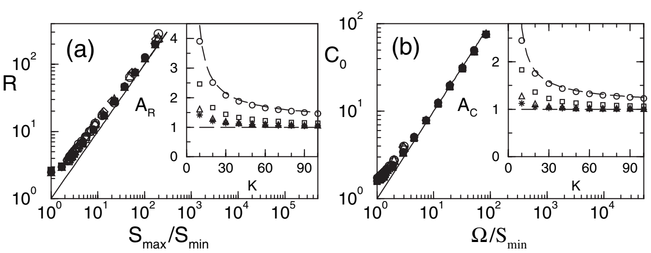

FIG. 1. (a) R as a function of $S_{\max}/S_{\min}$ and (b) C_0 as a function of $\Omega/S_{\min}$, averaged over 50 realizations of the networks. Solid symbols: uniform distribution of $S_i \in [S_{\min}, S_{\max}]$. Open symbols: power-law distribution of S_i, $P(S) \sim S^{-\Gamma}$ for $2.5 \leq \Gamma \leq 10$. Different symbols are for networks with different topologies: BA growing SFNs (○), growing SFNs with aging exponent $\alpha = -3$ (□), random SFNs with $\gamma = 3$ (◇), and K-regular random networks (△). The number of nodes is $N = 2^{10}$ and the mean degree is $K = 20$. Insets of (a) and (b): A_R and A_C as functions of K for $S_{\max}/S_{\min} = 1$ (○), 2 (□), 10 (△), and 100 (*), obtained with uniform distribution of S_i in K-regular networks. The dashed lines are the bounds. Solid lines in (a) and (b): Eq. (13) with $A_R = A_C = 1$.

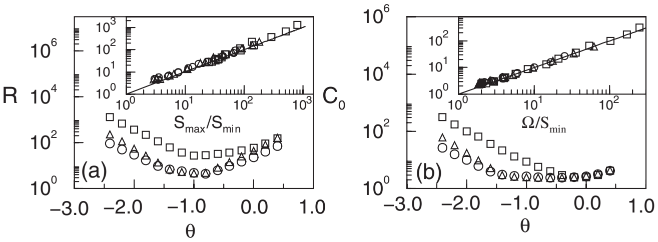

FIG. 2. (a) Eigenratio R and (b) cost C_0 as functions of θ for the BA growing SFNs (○), growing SFNs with aging exponent $\alpha = -3$ (□), and random SFNs with $\gamma = 3$ (△). Each symbol is an average over 50 realizations of the networks with $K = 20$ and $N = 2^{10}$. Inset of (a): the same data for R as a function of S_{max}/S_{min}. Inset of (b): the same data for C_0 as a function of Ω/S_{min}. Solid lines: Eq. (13) with $A_R = A_C = 1$.

weighted SFNs, the eigenratio R as a function of the network size N collapses to a single universal curve when plotted against S_{max}/S_{min} [22].

In addition to heterogeneous degrees and weights, many real networks also display high clustering [23] and nontrivial correlation of degrees [24]. The clustering and degree correlations are negligible in random SFNs and K-regular random networks, but we find that they are significant in growing SFNs for $\alpha < 0$. The results in Figs. 1–3 show that the universality holds without significant dependences on these topological properties.

Equation (13) also provides a meaningful approximation for networks which are not fully random. For example, consider small-world networks [23] where a regular ring of N $(= 2^{10})$ nodes, each connected to K $(= 20)$ nearest neighbors, is rewired with a probability p for each link. We find that R and C_0 collapse to the universal curves even when the networks are dominated by local connections, e.g., for $p = 0.3$, if the intensities are very heterogeneous ($S_{max}/S_{min} \geq 10$). For networks with $k_{min} \sim 1$, the synchronizability is still strongly dependent on S_{max}/S_{min} and K, although it shows additional dependences on the details of the distributions of S_i and k_i and on other topological properties.

In summary, we have shown that the synchronizability of sufficiently random networks with minimum degree $k_{min} \gg 1$ is universally dominantly determined by the mean degree K and the heterogeneity of the intensities S_i. This universality applies to a general class of large networks where the heterogeneity of S_i is due to either the distribution of degrees, as in unweighted SFNs, or the distribution of connection weights, as in weighted K-regular networks, or a combination of both, as expected in most realistic networks, such as in the airport network [13], which underlies the synchronization of epidemic outbreaks [11]. In particular, formula (13) explains why synchronizability is improved when the heterogeneity of S_i is reduced, which can be useful for network design and control of synchronization.

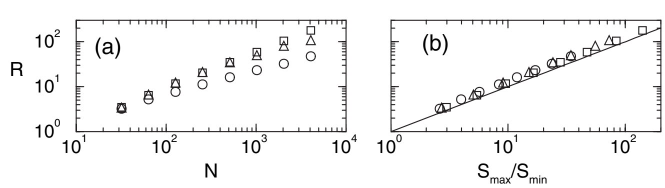

FIG. 3. (a) R as a function of the system size N for unweighted networks ($\theta = 0$). (b) The same data for R as a function of S_{max}/S_{min}. Solid line: Eq. (13) with $A_R = 1$. The other parameters and legends are the same as in Fig. 2(a).

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