The chaotic channel

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This work combines the Theory of Chaotic Synchronization with the Theory of Information in order to introduce the chaotic channel, an active media formed by connected chaotic systems. This subset of a large chaotic net represents the path along which information flows. We show that the possible amount of information exchange between the transmitter, where information enters the net, and the receiver, the destination of the information, is proportional to the level of synchronization between these two special subsystems. Another important foretelling of this approach is that if the receiver and the transmitter are phase or completely synchronized, transmission has the capacity to happen with no errors.

A communication system, as defined by Shannon [1], is composed of an **information source** that produces a message, a **transmitter**, that transforms the message into a signal suitable for transmission over a **channel**, such that the message can be retrieved in the **receiver** with a minimal amount of errors. The most outstanding result in Shannon's Theory of Communication is the formula that gives the channel capacity, i.e., the average upper bound for the mutual information exchange between the transmitter and the receiver, or in other words, the possible amount of information that can be transmitted in a physical medium.

In chaos-based communication systems, each step of the communication can be performed using a chaotic system. As shown in [2], chaotic systems can naturally possess the properties of a transmitter, since they can be controlled such that the information of the message is encoded in its chaotic trajectory. Moreover, a chaotic trajectory is suitable for transmission over noisy and frequency band-limited channels, in the sense that the receiver can recover the message with a small amount of errors [3–6].

A channel as defined by Shannon is a physical medium that enables information to pass throughout until it arrives to the receiver. Analogously, we define a chaotic channel as an active physical medium formed by at least two connected chaotic systems that enable information from a source to pass from the first one (the transmitter) to the last one (the receiver). A chaotic net, formed by many connected elements might possess only a few chaotic channels, in the sense that the channel is the path of connected systems along which information flows. We define a transmitter and a receiver in this net to be both elected subsystems of the whole chaotic net.

A first step to understand the chaotic channel goes back to the works [7–9] in which it is shown that two coupled chaotic systems can become Completely Synchronized (CS), i.e., the distance between their initially different trajectories tends to zero, as time tends to infinity. This property was explored as a communication system, making the pair of coupled systems to work as an active media that transports information from a driving system (the transmitter) to a slave system (the receiver) [8, 9]. The condition under which CS takes place is given by the conditional exponents [8]. Basically, two coupled chaotic systems have two sets of conditional exponents. One set is associated with the synchronization manifold and the other one associated with the transversal manifold. The presence of positive transversal exponents indicate that CS does not exist.

A second step is given by [10, 11]. A chaotic trajectory produces to an observer a certain amount of uncertainty, that defines information, quantified by the Kolmogorov-Sinai entropy H_{KS} [10], which is the proper way of calculating the Shannon source entropy of a chaotic set. For systems with a measurable (the trajectory is bounded to a finite domain) and ergodic (average quantities can be calculated in space and time) invariant (with respect to time translations of the system and to smooth transformations) natural measure, that is smooth along the unstable manifold, H_{KS} equals the sum of the positive Lyapunov exponents [11]. So, as a source of information, the more chaotic a system is, the more information it produces.

A third step is given in [12], which showed that the conditional exponents alike the Lyapunov exponents are relevant physical quantities to describe a network that is formed by coupled chaotic systems. In particular, in addition to Pesin's identity [11], it was suggested [12] that the summation of the positive conditional exponents λ^+ between two subsystem of a large network could be a measure of the apparent rate of information production in each pair of subsystems, as if they were detached from the whole group.

We show in this paper, by plausible physical reasonings, that the appropriate quantity to quantify the amount of information in the chaotic channel is

$$I_C = \sum \lambda_{\parallel}^{+} - \sum \lambda_{\perp}^{+}, \tag{1}$$

where I_C represents the mutual information between the transmitter and the receiver, $\sum \lambda_{\parallel}^{+}$, the sum of the positive exponents associated with the synchronization manifold, and $\sum \lambda_{\perp}^{+}$, the sum of the positive exponents associated to the transversal manifold. The term $\sum \lambda_{\parallel}^{+}$ represents the information (entropy production per time unit) produced by the synchronous trajectories, and it corresponds to the amount of information transmitted.

The term $\sum \lambda_{\perp}^{+}$ represents the information produced by the non-synchronous trajectories, and it corresponds to the information lost in the transmission, the information that is erroneously retrieved in the receiver. From Shannon's work, the mutual information between the transmitter, S_1, and the receiver, S_2, is given in a colloquial term as the amount of information transmitted minus the information lost due to errors. So, it is suggestive to assume that in coupled chaotic systems, the mutual information is given by Eq. (1).

Finally, the capacity as defined in Shannon's work, is the maximum of the mutual information. So, while the capacity of a net that respects certain conditions [11] is given by $H_{KS} = \sum_{\lambda_k > 0} \lambda_k$, with λ_k representing all the possible Lyapunov exponents of the net, where a chaotic channel is located, the capacity of the chaotic channel is given by $C = \max(I_C)$, with the condition that

$$H_{KS} \geq C. \tag{2}$$

We implement this approach for a system of two coupled maps, and for a system of three coupled Rössler oscillators, showing that this approach is valid for both descriptions of dynamical systems, the discrete and the time-continuous. Further and in the conclusions, we argue that these results can be extended to large networks of coupled chaotic systems, as well.

The discrete channel - *a channel of communication formed by discrete chaotic elements:* We model a discrete channel by two coupled maps $x^{(1)}_{n+1} = (1-c)2x^{(1)}_n + 2cx^{(2)}_n$ (mod 1) and $x^{(2)}_{n+1} = (1-c)2x^{(2)}_n + 2cx^{(1)}_n$ (mod 1), with $c \leq 0.5$, representing the coupling strength. In here, the channel is completely described only by the transmitter, the subsystem of variable $x^{(1)}$, and the receiver, the subsystem of variable $x^{(2)}$. The Lyapunov exponents of these coupled systems are λ_1=2 and λ_2=2-4c. Therefore, H_{KS} = 4-4c. The synchronization manifold, $x_{\|}$, is defined by the following variable transformation: $x_{\|} = x^{(1)} + x^{(2)}$, and the transversal manifold is defined by $x_{\perp} = x^{(1)} - x^{(2)}$. The conditional exponents are $\lambda_{\|}$=2 and $\lambda_{\perp}$=2-4c, therefore, equal to the Lyapunov exponents. For no coupling (c=0), these two mappings work as independent sources of information, and the capacity for generating information of these two sources are given by the sum of the capacity of each one, which in this case is equal to $H_{KS} = \lambda_1 + \lambda_2 = 4$. The mutual information should vanish (note that for c=0, $\lambda_{\|} - \lambda_{\perp}$=0) with the errors produced by the non-synchronous trajectories being maxima (note that $\lambda_{\perp}$ is maxima for c=0). This I_C function increases as the coupling c increases, once the larger is c, the larger is the synchronization level, and consequently the amount of information retrieved in the receiver. So, we see that it is reasonable to consider that $I_C = \lambda_{\|} - \lambda_{\perp}$. Note that Eq. (2) holds, once $H_{KS} = 4 - 4c \geq I_C$, with I_C= 4c. We get equality for c=1/2 (H_{KS}=I_C) when CS is reached between the transmitter and receiver. At this moment, the errors produced by the non-synchronous trajectories should vanish. That is exactly what happens to $\lambda_{\perp}$. Therefore, we see again that it is reasonable to consider that $\lambda_{\perp}$ is related to the errors caused by the non-synchronous trajectories in the decoding of the information in the receiver. So, when there is no CS, errors may occur in the transmission ($\sum \lambda_1^{+} > 0$), while when there is CS, errors may not occur and the channel transmits information in its full capacity.

The continuous channel: - *a channel of communication formed by continuous chaotic elements* A small chaotic network is modeled by the following system of three coupled Rössler oscillators: $\dot{x}_i = -\alpha_i y_i - z_i + A_{ji}(x_j - x_i)$, $\dot{y}_i = \alpha_i x_i + a y_i$, $\dot{z}_i = b + z_i(x_i - c)$, with a=0.15, b=0.2 and c=10, and $i, j = 1, \ldots, 3$. The index i and j denotes systems S_i and S_j. A_{ji} indicates the coupling of the perturbation that S_j makes in S_i. We use the index i to represent the transmitter and the index j to represent the receiver. The configuration of the net is set to have S_1 and S_2 bidirectionally coupled with A_{12}=A_{21}, and S_3 is unidirectionally coupled to S_2, that is, $A_{23} \geq$0 and A_{32}=0. α_1=1, α_2=1.0002, and α_3=0.998, and thus all the systems have different parameters.

The absence of non-local connections (A_{13}=A_{31}=0) is imposed such that we are able to calculate the capacity of the chaotic channel only by looking at pairs of nearby elements. So, for this network of three elements, instead of looking at the mutual information between S_1 and S_3 (whose calculation involves terms on the subspaces of all the other subsystems of the net), we look at the mutual information between S_1 and S_2, and also between S_2 and S_3, and infer what is the mutual information between S_1 and S_3. This assumption simplifies enormously the analytical (and numerical) calculation of the conditional exponents.

Assuming $\vec{X}_i$ to describe the state variables of subsystem i, then the synchronization manifold between subsystem S_i and S_j is given by $x^{ij}_{\|} = \vec{X}_i + \vec{X}_j$, which yields the ODE's that describe this manifold. $\dot{x}^{ij}_{\|} = [(\alpha_j - \alpha_i)y_{\perp} - (\alpha_i + \alpha_j)y_{\|}]/2 - z_{\|}\ G_{\|} +$, $\dot{y}^{ij}_{\|} = [(\alpha_i + \alpha_j)x_{\|} + (\alpha_i - \alpha_j)x_{\perp}]/2 + a y_{\|}$, $\dot{z}^{ij}_{\perp} = 2b + (0.5x_{\|} - c)z_{\|} + 0.5x_{\perp}z_{\perp}$. The transversal manifold is defined as $x^{ij}_{\perp} = \vec{X}_i - \vec{X}_j$, which give us $\dot{x}^{ij}_{\perp} = [(\alpha_j - \alpha_i)y_{\|} - (\alpha_i + \alpha_j)y_{\perp}]/2 - z_{\perp} + G^{ij}_{\perp}$, $\dot{y}^{ij}_{\perp} = [(\alpha_i + \alpha_j)x_{\perp} + (\alpha_i - \alpha_j)x_{\|}]/2 + a y_{\perp}$, $\dot{z}^{ij}_{\perp}$= $0.5x_{\perp}z_{\|} + (0.5x_{\|} - c)z_{\perp}$, with the term $G_{\perp}$ and $G_{\|}$ expressing the coupling between the transmitter and the receiver, with other elements in the network. The Jacobian of the whole net in the partial derivatives $\partial/\partial x^{ij}_{\|}$ and $\partial/\partial x^{ij}_{\perp}$ is formed by a 18 × 18 matrix. The conditional exponents are the logarithms of the eigenvalues of the product of these Jacobian matrix, as we iterate the trajectory. Assuming that terms given by the product of exclusively variables in the transversal coordinates do not contribute to the diagonal of the Jacobian product, as well as, are irrelevant in comparison with terms exclusively given by the product of the synchronization variables, the Jacobian in the variables $x^{ij}_{\|}$ is exclusively dependent only on terms that depend on the variables $x^{ij}_{\|}$ (the same happens for the variables $x^{ij}_{\perp}$), if i and j

are neighbors. That is, $i = 1$ and j=2, or i=2, or j=3. This means that the transversal exponents do not depend on the parallel exponents, and vice-versa, and they can be easily calculated by two matrices that have the same dimensionability as the chaotic subsystems (3×3). This property, a consequence of the linearity of the coupling, does not hold for non-local couplings with terms given by i=1, j=3. If this is the case, our approach still can be used, but one has to calculate a matrix of size 18×18. Also, for this proposed linear and only local couplings, each positive Lyapunov exponent can be associated exclusively to the capacity of one subsystem in the net, i.e., $H(S_i)$.

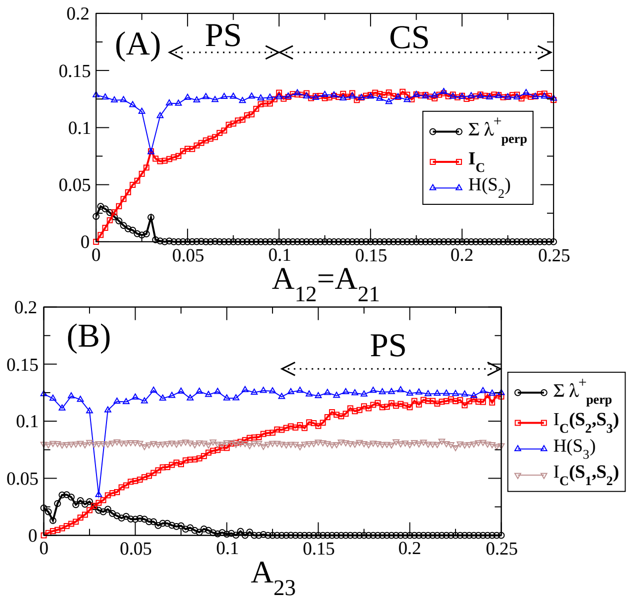

FIG. 1: [Color online] The mutual information I_C between receiver and transmitter in the network formed by three coupled Rössler systems. We show the information lost in the transmission ($\sum \lambda_\perp^+$), and the information retrieved by the transmitter, the mutual information ($\sum \lambda_\parallel^+$). In (A) the transmitter is S_1 and in (B), we can consider as transmitters (receivers) S_1 or S_2 (S_2 or S_3), and A_{12}=A_{21}=0.05. CS between S_2 and S_3 is achieved for $A_{23} \geq$0.31. The units are in bits per time unit. The intermittency present in the transition to PS in (A) causes numerical errors that apparently makes $I_C > H(S_2)$.

In Fig. 1 we show the formation of channels of communication, in this three elements net. In (A) we consider that there is no coupling between S_3 and S_2 (A_{23}=0), and therefore, $I_C(S_1, S_3)$=0. As we increase the coupling between S_1 and S_2, the mutual information $I_C(S_1, S_2)$ increases from 0 to $I_C(S_1, S_2) \cong$0.126 bits per time unit, the maximum value for the mutual information, that is the capacity of the chaotic channel. Note that the capacity of the net is given by $H_{KS} \cong 0.26$. Two phenomena are important to characterize the chaotic channel: (i) First, the appearance of Phase Synchronization (PS) [13] between S_1 (transmitter) and S_2 (receiver). In this phenomenom the amplitudes of the two systems are uncorrelated while their phase difference remains bounded. Whenever that happens $\sum \lambda_\perp \to 0$ (arbitrarily close to zero), and therefore, the error in the retrieving of information in the receiver, caused by the non-synchronous trajectories, is already a minimum, i.e. there is a large chance that the message is completely recovered, with no errors, or with an insignificant small amount of errors, as discussed in [6]; (ii) The appearance of CS, makes $\sum \lambda_\perp$ =0, which means that the message can be completely recovered, with no errors, with the extra fact that the channel has its maximal capacity, i.e., $I_C(S_1, S_2)$ = C. Then, we fix the coupling between S_1 and S_2 (A_{12}=A_{21}=0.05), so to have PS between S_1 and S_2, and increase the coupling between S_2 and S_3. These three coupled systems can be treated as forming two communication channels. One from S_1 to S_2 and another from S_2 to S_3. Assuming, S_1 to be the transmitter and S_2 the receiver, $I_C(S_1, S_2) \cong$0.078. As we increase A_{23}, the channel formed by S_2 and S_3 has the same characteristics as shown in (A), that is, when S_2 and S_3 presents PS, $\sum \lambda_\perp \to 0$, and when S_2 and S_3 are in CS (what happens for $A_{23} \geq$0.31), $\sum \lambda_\perp$=0. Thus, the mutual information between the systems S_1 and S_3 is therefore given by the minimum between $I_C(S_1, S_2)$ and $I_C(S_2, S_3)$, and the capacity of this channel can be achieved for a set of couplings such that the minimum between $I_C(S_1, S_2)$ and $I_C(S_2, S_3)$ is a maximum. Consequently, that happens when S_1, S_2, and S_3 are in CS. In this figure, we also plot the information produced by the receiver $H(S_2)$, in (A) and the receiver $H(S_3)$ in (B). At the moment the receiver is completely synchronized with the transmitter, the information of the receiver equals the mutual information, i.e., $H(S_2)$=$I_C(S_1, S_2)$ in (A), and $H(S_3)$=$I_C(S_2, S_3)$ for $A_{23} \geq$0.31. This result, clearly demonstrates that the possible amount of information that can be transmitted to the net is fundamentally bounded by the capacity of the receiver to generate information. We have extended these studies numerically to large networks and get analogous results. Next, we make general remarks concerning this approach.

Using the chaotic channel as a medium to transport information: Whether or not a stimulus propagates along the net depends on how is the iteration between the channel and the stimulus. In principle, the more synchronous is the network to the stimulus, the more information is transmitted. In a general situation, we expect the stimulus to be complex and to be not completely synchronized to the network, once we expect the network to have a different dynamical character than the stimulus. Yet, we expect that some information about the stimulus is transmitted. This would only be possible if some quantity from the stimulus synchronizes with the network. For example, the average frequency, as in a chaotic system that is phase synchronized with a periodic/chaotic stimulus. At this situation, information about the average frequency can be retrieved some-where (and somehow) in the chaotic channel. Assuming that the stimulus does not alter basic properties of the Jacobian matrix of the synchronization and transversal variables (one situation is when the stimulus strength is small), the maximum possible amount of information retrieved about the stimulus would still be given by Eq. (1). In other words, the capacity of the chaotic channel limits the amount of information that can be retrieved about the stimulus.

The noisy chaotic channel and the recovery of information: H_{KS} is a measure of uncertainty about the forward time evolution of the trajectory realized up to some precision, when a series of previous observations with the same precision had been already performed. It does not reflect the amount of information retrieved from one particular observation, realized with some specified precision. In order to understand how much information can be withdrawn from one single observation in a chaotic system, the accuracy with which this observation is realized determines this amount of information, which is a multiple of H_{KS} [4]. From [4], we have that each observation realized in a one-dimensional map provides $(g+1)H_{KS}$ bits. g is an integer number that is proportional to the accuracy of the observation and inversely proportional to the amount of noise in the chaotic trajectory. Using the deterministic property of chaotic systems, each observed trajectory point generates more g other trajectories points, that were not observed. For the continuous chaotic channel, each observation of the trajectory on a Poincaré plane can be used to reveal the other g non-observed crossings of the trajectory in this same plane [14]. So, each observation gives R bits of information, with $R = (g+1)I_C(S_i, S_j) \times \langle T(S_2) \rangle$ bits, and g being an integer number proportional to the accuracy of the observation (inversely proportional to the noise variance), with $\langle T(S_2) \rangle$ being defined in Ref. [14]. Note that the average time interval to obtain all this information is equal to $(g+1)\langle T(S_2) \rangle$, and therefore, the rate at which one recovers information in the receiver ($\frac{R}{(g+1)\langle T(S_2) \rangle}$) is at most equal to the rate of information produced in the transmitter (I_C) [15].

Concluding, we define the chaotic channel as a subset of a net of coupled chaotic systems, along which information flows. We characterize this channel by showing how to calculate the amount of information interchange between two important elements of the channel: the **transmitter**, which can be thought as an entrance door of the information in the net, and the **receiver**, the ending point of the information. From the whole capacity of the net, given by the summation of all the positive Lyapunov exponents for the whole net, the channel, along which information flows, has a much more limited capacity. A small coupling between the elements of the channel means that information can already be exchanged between the transmitter and receiver. If phase synchronization exists between the elements of the channel, a transmitted message can be fully recovered at a rate smaller than if these elements are fully synchronized, a situation for which the channel achieves its capacity. In the case one wants to study large networks in which not only local but also any sort of non-local coupling is possible, our method can still be used. As an example, for a network of N systems, each one having dimension D, one would have to construct a matrix of size $2D.M \times 2D.M$, where M is the number of possible different connections between the systems. For example, if N=4, thus M=6, representing the number of possible connections between all the elements of the network: S_1 with S_2, S_1 with S_3, S_1 with S_4, S_2 with S_3, S_2 with S_4, and S_3 with S_4. So, in principle, one can use this approach to understand information transmission in more complex systems, as natural chaotic nets, e.g. the Human brain, which shows evidence of chaotic behavior [18], ecological systems [20], coupled lasers [21], ensembles of neuron model oscillators [22]. Also, for transient dynamics, as in the brain of birds [19], in the case such dynamics could be governed by an asymptotic chaotic set from which I_C is calculated.

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[14] I_C in Eq. (1) has the units of bits per time unit. It would

be advantageous to calculate it in bits per cycle, that is, $I'_C = I_C \times \langle T(S_l) \rangle$, where $\langle T(S_l) \rangle$ is the average time of the returnings of the system S_l, i.e., $\langle T \rangle = \frac{\sum_{k=1}^{R_l} \tau_k(S_l)}{R_l}$, with $\tau_K(S_l)$, a series of time intervals for the trajectory to leave and to return the **Poincaré plane** positioned at y_l=0. Recall that this can be done since the conditional exponents are time invariant, as well as the normal Lyapunov exponents.

[15] As already discussed in Ref. [2, 5], a chaotic system can encode information at a rate given by the Topological entropy [16], H_T, which, as shown in [5], $H_T \geq H_{KS}$. Therefore, chaotic systems, as transmitters, can encode more information than the one given by H_{KS}. However, not all this information can be retrieved in the receiver, otherwise, we would have an apparent violation of the Shannon Communication Theory, as it was shown in Ref. [17]. Thus, the correct amount of information that can be retrieved in the receiver is given by I_C.

[16] Given H_T to be the topological entropy, and, M, the number of periodic orbits of period P present in an attractor, $M \propto \exp(H_T \times P)$.

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