Network synchronization, diffusion, and the paradox of heterogeneity

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Many complex networks display strong heterogeneity in the degree (connectivity) distribution. Heterogeneity in the degree distribution often reduces the average distance between nodes but, paradoxically, may suppress synchronization in networks of oscillators coupled symmetrically with uniform coupling strength. Here we offer a solution to this apparent paradox. Our analysis is partially based on the identification of a diffusive process underlying the communication between oscillators and reveals a striking relation between this process and the condition for the linear stability of the synchronized states. We show that, for a given degree distribution, the maximum synchronizability is achieved when the network of couplings is weighted and directed and the overall cost involved in the couplings is minimum. This enhanced synchronizability is solely determined by the mean degree and does not depend on the degree distribution and system size. Numerical verification of the main results is provided for representative classes of small-world and scale-free networks.

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I. INTRODUCTION

The interplay between network structure and dynamics has attracted a great deal of attention in connection with a variety of processes [1], including epidemic spreading [2], congestion and cascading failures [3], and synchronization of coupled oscillators [4–11]. Much of this interest has been prompted by the discovery that numerous real-world networks [1] share universal structural features, such as the small-world [12] and scale-free properties [13]. Small-world networks (SWN's) exhibit a small average distance between nodes and high clustering [12]. Scale-free networks (SFN's) are characterized by an algebraic, highly heterogeneous distribution of degrees (number of links per node) [13]. Most SFN's also exhibit a small average distance between nodes [14] and this distance may become smaller as the heterogeneity (variance) of the degree distribution is increased [15]. It has been shown that these structural properties strongly influence the dynamics on the network.

In oscillator networks, the ability to synchronize is generally enhanced in both SWN's and random SFN's as compared to regular lattices [16]. However, it was recently shown that random networks with strong heterogeneity in the degree distribution, such as random SFN's, are much more difficult to synchronize than random homogeneous networks [8], even though the former display smaller average distance between nodes [15]. This result is interesting for two main reasons. First, it challenges previous interpretations that the enhancement of synchronizability in SWN's and SFN's would be due to the reduction of the average distance between oscillators. Second, in networks where synchronization is desirable, it puts in check the hypothesis that the scale-free property has been favored by evolution for being dynamically advantageous.

Previous work has focused mainly on the role played by shortest paths between nodes. By considering only shortest paths it is implicitly assumed that the information spreads only along them. However, the communication between oscillators is more closely related to a process of diffusion on the network, which is a process involving all possible paths between nodes. Another basic assumption of previous work is that the oscillators are coupled symmetrically and with the same coupling strength [8]. Under the assumption of symmetric coupling, the maximum synchronizability may be indeed achieved when the coupling strength is uniform [9]. But to get a better synchronizability, the couplings are not necessarily symmetrical. Many real-world networks are actually directed [1] and weighed [17], and the communication capacity of a node is likely to saturate when the degree becomes large.

In this paper, we study the effect that asymmetry and saturation of coupling strength have on the synchronizability of complex networks. We identify a physical process of information diffusion that is relevant for the communication between oscillators and we investigate the relation between this process and the stability of synchronized states in directed networks with weighted couplings. We address these fundamental issues using as a paradigm the problem of complete synchronization of identical oscillators.

We find that the synchronizability is explicitly related to the mixing rate of the underlying diffusive process. For a given degree distribution, the synchronizability is maximum when the diffusion has a uniform stationary state, which in general requires the network of couplings to be weighted and directed. For large sufficiently random networks, the maximum synchronizability is primarily determined by the mean degree of the network and does not depend on the degree distribution and system size, in sharp contrast with the case of unweighted (symmetric) coupling, where the synchronizability is strongly suppressed as the heterogeneity or number of oscillators is increased. Furthermore, we show that the total cost involved in the network coupling is significantly reduced, as compared to the case of unweighted coupling,

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and is minimum when the synchronizability is maximum. Some of these results were announced in Ref. [18].

The fact that the communication between oscillators takes place along all paths explains why the synchronizability does not necessarily correlate with the average distance between oscillators. Moreover, the synchronizability of SFN's is strongly enhanced when the network of couplings is suitably weighted. This, in addition to the well-known improved structural robustness of SFN's [19], may have played a crucial role in the evolution of many SFN's.

The paper is organized as follows. In Sec. II, we introduce the synchronization model and the measure of synchronizability. In Sec. III, we study the corresponding process of diffusion. In Sec. IV, we focus on the case of maximum synchronizability. The problem of cost is considered in Sec. V. In Sec. VI, we present direct simulations on networks of maps. Discussion and conclusions are presented in the last section.

II. FORMULATION OF THE PROBLEM

We introduce a generic model of coupled oscillators and we present a condition for the linear stability of the synchronized states in terms of the eigenvalues of the coupling matrix.

A. Synchronization model

We consider complete synchronization of linearly coupled identical oscillators:

$$\frac{dx_i}{dt} = f(x_i) - \sigma \sum_{j=1}^{N} G_{ij} h(x_j), \quad i = 1,2, \ldots N, \tag{1}$$

where $f=f(x)$ describes the dynamics of each individual oscillator, $h=h(x)$ is the output function, $G=(G_{ij})$ is the coupling matrix, and σ is the overall coupling strength. The rows of matrix G are assumed to have zero sum to ensure that the synchronized state $\{x_i(t)=s(t), \forall i \,|\, ds/dt=f(s)\}$ is a solution of Eq. (1).

In the case of symmetrically coupled oscillators with uniform coupling strength, G is the usual (symmetric) Laplacian matrix $L=(L_{ij})$: the diagonal entries are $L_{ii}=k_i$, where k_i is the degree of node i, and the off-diagonal entries are $L_{ij}=-1$ if nodes i and j are connected and $L_{ij}=0$ otherwise. For $G_{ij}=L_{ij}$, heterogeneity in the degree distribution suppresses synchronization in important classes of networks [8] (see also Ref. [10]). The synchronizability can be easily enhanced if we modify the topology of the network of couplings. Here, however, we address the problem of enhancement of synchronizability for a *given* network topology.

In order to enhance the synchronizability of heterogeneous networks, we propose to scale the coupling strength by a function of the degree of the nodes. For specificity, we take

$$G_{ij} = L_{ij}/k_i^{\beta}, \tag{2}$$

where β is a tunable parameter. We say that the network or coupling is weighted when $\beta \neq 0$ and unweighted when $\beta = 0$. The underlying network associated with the Laplacian matrix L is undirected and unweighted, but for $\beta \neq 0$, the network of couplings becomes not only weighted but also directed because the resulting matrix G is in general asymmetric. This is a special kind of directed network where the number of *in*-links is equal to the number of *out*-links in each node, and the directions are encoded in the strengths of in- and out-links. In spite of the possible asymmetry of matrix G, all the eigenvalues of matrix G are nonnegative reals and can be ordered as $0=\lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_N$, as shown below.

B. Basic spectral properties

Equation (2) can be written as

$$G = D^{-\beta} L, \tag{3}$$

where $D=\mathrm{diag}\{k_1, k_2, \ldots, k_N\}$ is the diagonal matrix of degrees. (We recall that the degree k_i is the number of oscillators coupled to oscillator i.) From the identity $\det(D^{-\beta}L - \lambda I) = \det(D^{-\beta/2} L D^{-\beta/2} - \lambda I)$, valid for any λ, where "det" denotes the determinant and I is the $N \times N$ identity matrix, we have that the spectrum of eigenvalues of matrix G is equal to the spectrum of a symmetric matrix defined as

$$H = D^{-\beta/2} L D^{-\beta/2}. \tag{4}$$

That is, $\rho(G)=\rho(H)$, where ρ denotes the set of eigenvalues. From this follows that all eigenvalues of matrix G are real, as anticipated above. It is worth mentioning that, although the eigenvalues of G and H are equal, from the numerical point of view it is much more efficient to compute the eigenvalues from the symmetric matrix H than from G.

Additionally, all the eigenvalues of matrix G are nonnegative because H is positive semidefinite, and the smallest eigenvalue λ_1 is always zero because the rows of G have zero sum. Moreover, if the network is connected, then $\lambda_2 > 0$ for any finite β. This follows from the corresponding property for L and Eq. (4)—i.e., the fact that matrices H and L are congruent. Naturally, the study of complete synchronization of the whole network only makes sense if the network is connected.

For $\beta=1$, matrix H is the normalized Laplacian matrix studied in spectral graph theory [20]. In this case, if $N \geqslant 2$ and the network is connected, then $0 < \lambda_2 \leqslant N/(N-1)$ and $N/(N-1) \leqslant \lambda_N \leqslant 2$. For spectral properties of unweighted SFN's, see Refs. [21–23].

C. Synchronizability

The variational equations governing the linear stability of a synchronized state $\{x_i(t)=s(t), \forall i\}$ of the system in Eqs. (1) and (2) can be diagonalized into N blocks of the form

$$\frac{d\eta}{dt} = [Df(s) - \alpha Dh(s)]\eta, \tag{5}$$

where D denotes the Jacobian matrix, $\alpha=\sigma\lambda_i$, and λ_i are the eigenvalues of the coupling matrix G. The largest Lyapunov exponent $\Gamma(\alpha)$ of this equation can be regarded as a master stability function, which determines the linear stability of the synchronized state for any linear coupling scheme [24]: the synchronized state is stable if $\Gamma(\sigma\lambda_i)<0$ for $i=2,\ldots,N$. (The eigenvalue λ_1 corresponds to a mode parallel to the synchronization manifold.)

For many widely studied oscillatory systems [7,24], the master stability function $\Gamma(\alpha)$ is negative in a single, finite interval (α_1,α_2). Therefore, the network is synchronizable for some σ when the eigenratio $R=\lambda_N/\lambda_2$ satisfies

$$R<\alpha_2/\alpha_1. \tag{6}$$

The right-hand side of this equation depends only on the dynamics (f, h, and s), while the eigenratio R depends only on the coupling matrix G. The problem of synchronization is then reduced to the analysis of eigenvalues of the coupling matrix [7]: the smaller the eigenratio R, the larger the synchronizability of the network and vice versa.

III. DIFFUSION AND BALANCE OF HETEROGENEITY

We study a process of diffusion relevant for the communication between oscillators and we argue that the synchronizability is maximum (R is minimum) for $\beta=1$.

A. Diffusion process

From the identity

$$\sum_{j=1}^{N} G_{ij}h(x_j)=\sum_{j=1}^{N} k_i^{-\beta}A_{ij}[h(x_i)-h(x_j)], \tag{7}$$

we observe that the weighted coupling scheme in Eqs. (1) and (2) is naturally related to a diffusive process with absorption and emission described by the transition matrix

$$P=\frac{1}{\Lambda}D^{-\beta}A, \tag{8}$$

where $A=D-L$ is the adjacency matrix and Λ is the largest eigenvalue of $D^{-\beta}A$. According to this process, if we start with an arbitrary distribution $y=(y^{(1)},\ldots,y^{(N)})$, where $y^{(i)}$ is associated with the initial state at node i, after n time steps the distribution is $P^n y$. This process is different from the usual (conservative) random walk process. In particular, because $\Sigma_i P_{ij}$ may be different from 1, the diffusion *in* and *out* of a node may differ even in the stationary state.

For instance, consider a network of three nodes, a-b-c, where nodes a and c have degree 1 and are both connected to node b. For $\beta=1$, the transition matrix is

$$P=\begin{pmatrix} 0 & 1 & 0\\ 1/2 & 0 & 1/2\\ 0 & 1 & 0\end{pmatrix}. \tag{9}$$

In the stationary state, each of the three nodes has, say, one unity (of the "diffusive quantity"). At each time step, node b receives 1/2 unit from node a and 1/2 unit from node c, and each of the nodes a and c receives 1 unit from node b. Therefore, node b sends a total of 2 units and receives only 1 unit, while each of the nodes a and c sends 1/2 unit and receives 1 unit. This means that there is an "absorption" of 1/2 unit at each of the nodes a and c and the "emission" of 1 unit at node b. It is in this sense that the matrix in Eq. (8) describes a diffusive process with absorption and emission.

On the other hand, the usual random walk process is conservative at each node. Such a process is described by the matrix $D^{-\beta}AC^{-1}$, where $C_{ij}=\delta_{ij}\Sigma_{\ell\sim j}k_\ell^{-\beta}$ and the sum is over all the k_j nodes connected to node j. For a node i connected to a node j and a uniform distribution, this conservation law implies that the amount of information that node i receives from node j depends on the degree of all the nodes connected to node j. But this is not what happens in a network of self-sustained oscillators. In a network of oscillators, the amount of information that oscillator i receives directly from oscillator j can only depend on the strength of the coupling from j to i, which is proportional to $1/k_i^{\beta}$, as in the process described by the transition matrix in Eq. (8).

B. Balance of heterogeneity

Because the master stability function $\Gamma(\alpha)$ is negative in a finite interval (α_1,α_2), increasing (decreasing) the overall coupling strength σ beyond a critical value $\sigma_{max}(\sigma_{min})$ destabilizes the synchronized state. Dynamically, the loss of stability is due to a short- (long-) wavelength bifurcation at $\sigma=\sigma_{max}(\sigma_{min})$ [25] (see also Ref. [11]). Physically, this bifurcation excites the shortest (longest) spatial wavelength mode because some oscillators are too strongly (weakly) influenced by the others.

Now, consider the process of diffusion described by matrix P on a network where not all the nodes have the same degree. Starting with an arbitrary distribution y, after n steps we have $P^n y$. If we require the (stationary) distribution for $n\to\infty$ to be uniform, we obtain $\beta=1$ because this is the only case where $y_0=(1,1,\ldots,1)$ is an eigenvector associated with the eigenvalue 1, which is the largest eigenvalue of matrix P. For $\beta<1$, the distribution is more heavily concentrated on nodes with large degree. For $\beta>1$, the concentration happens on nodes with small degree. Physically, this means that for both $\beta<1$ and $\beta>1$ some oscillators are more strongly influenced than others and the ability of the network to synchronize is limited by the least and most influenced oscillators: for small (large) σ the system is expected to undergo a long- (short-) wavelength bifurcation due to the least (most) influenced nodes, as explained above. We then expect the network to achieve maximum synchronizability at $\beta=1$.

In Fig. 1 we show the numerical verification of this hypothesis for various models of complex networks. The networks are built as follows [26].

(i) *Random SFN's* [27]. Each node is assigned to have a number $k_i\geq k_{min}$ of "half-links" according to the probability distribution $P(k)\sim k^{-\gamma}$, where γ is a scaling exponent and k_{min} is a constant integer. The network is generated by randomly connecting these half-links to form links, prohibiting self- and repeated links. In the limit $\gamma=\infty$, all nodes have the same degree $k=k_{min}$.

(ii) *Networks with expected scale-free sequence* [22]. The network is generated from a sequence $\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_N$, where $\tilde{k}_i\geq\tilde{k}_{min}$ follows the distribution $P(\tilde{k})\sim\tilde{k}^{-\gamma}$ and $max_i\tilde{k}_i^2<\Sigma_i\tilde{k}_i$. A link is then independently assigned to each pair of

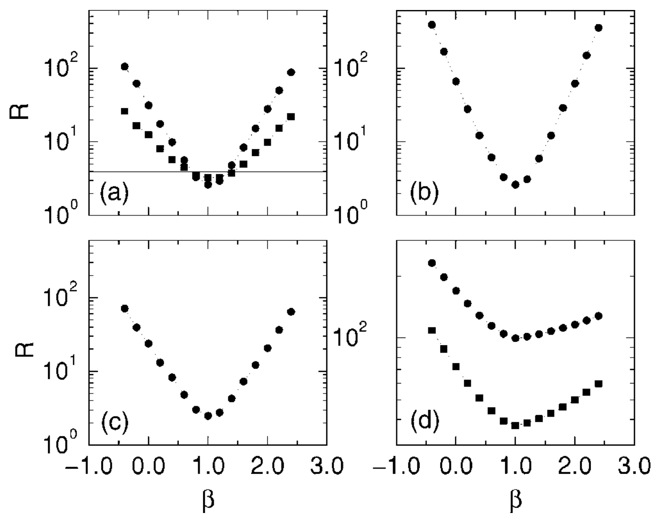

FIG. 1. Eigenratio R as function of β: (a) random SFN's with $\gamma=3$ (●), $\gamma=5$ (■), and $\gamma=\infty$ (solid line), for $k_{min}=10$; (b) networks with expected scale-free sequence for $\gamma=3$ and $\tilde{k}_{min}=10$; (c) growing SFN's for $\gamma=3$ and $m=10$; (d) SWN's with $M=256$ (●) and $M=512$ (■), for $\kappa=1$. Each curve is the result of an average over 50 realizations for $N=1024$.

nodes (i,j) with probability $p_{ij}=\tilde{k}_i\tilde{k}_j/\Sigma_i\tilde{k}_i$. In this model, self-links are allowed. We observe, however, that the eigenratio R is insensitive to the removal of self-links.

(iii) *Growing SFN's* [28]. We start with a fully connected network with m nodes and at each time step a new node with m links is added to the network. Each new link is connected to a node i in the network with probability $\Pi_i \sim (1-p)\hat{k}_i + p$, where $\hat{k}_i$ is the updated degree of node i and $0 \leqslant p \leqslant 1$ is a tunable parameter. For large degrees, the scaling exponent of the resulting network is $\gamma=3+p[m(1-p)]^{-1}$. For $p=0$, the exponent is $\gamma=3$ and we recover the Barabási-Albert model [13].

(iv) *SWN's* [29]. Starting with a ring of N nodes, where each node is connected to 2κ first neighbors, we add $M \leqslant N(N-2\kappa-1)/2$ new links between randomly chosen pairs of nodes. Self- and repeated links are avoided.

Our extensive numerical computation on models (i)–(iv) shows that the eigenratio R has a well defined minimum at $\beta=1$ in each case (Fig. 1). The only exception is the class of homogeneous networks, where all the nodes have the same degree k. When the network is homogeneous, the weights $k_i^{-\beta}$ can be factored out in Eq. (8) and a uniform stationary distribution is achieved for any β. In this case, the eigenratio R is independent of β, as shown in Fig. 1(a) for random homogeneous networks with $k=10$ (solid line). A random homogeneous network corresponds to a random SFN for $\gamma=\infty$. In all other cases, including the relatively homogeneous SWN's, the eigenratio exhibits a pronounced minimum at $\beta=1$ (note the logarithmic scale in Fig. 1).

In SWN's, the heterogeneity of the degree distribution increases as the number M of random links is increased. The eigenratio R at $\beta=0$ reduces as M is increased, but the eigenratio at $\beta=1$ reduces even more, so that the minimum of the eigenratio becomes more pronounced as the heterogeneity of the degree distribution is increased [Fig. 1(d)]. Similar results are observed in the original Watts-Strogatz model of SWN's, where the mean degree is kept fixed as the number of random links is increased [12]. SWN's of pulse oscillators also present enhanced synchronization at $\beta=1$ [6].

In SFN's, the heterogeneity increases as the scaling exponent γ is reduced. As shown in Fig. 1(a) for random SFN's, the minimum of the eigenratio R becomes more pronounced as the heterogeneity of the degree distribution is increased. The same tendency is observed across different models of networks. For example, for a given γ and $k_{min}=\tilde{k}_{min}$, the minimum of the eigenratio is more pronounced in networks with expected scale-free sequence [Fig. 1(b)] than in random SFN's [Fig. 1(a)], because the former may have nodes with degree smaller than $\tilde{k}_{min}$. For small γ, the eigenratio in growing SFN's [Fig. 1(c)] behaves similarly to the eigenratio in random SFN's [Fig. 1(a)]. A pronounced minimum for the eigenratio R at $\beta=1$ is also observed in various other models of complex networks [30].

C. Mean-field approximation

A mean-field approximation provides further insight into the effects of degree heterogeneity and the dependence of R on β.

The dynamical equations (1) can be rewritten as

$$\frac{dx_i}{dt} = f(x_i) + \sigma k_i^{1-\beta}[\overline{h_i} - h(x_i)], \tag{10}$$

where

$$\overline{h_i} = \frac{1}{k_i}\sum_j A_{ij}h(x_j) \tag{11}$$

is the local mean field from all the nearest neighbors of oscillator i. If the network is sufficiently random and the system is close to the synchronized state s, we may assume that $\overline{h_i} \approx h(s)$ and we may approximate Eq. (10) as

$$\frac{dx_i}{dt} = f(x_i) + \sigma k_i^{1-\beta}[h(s) - h(x_i)], \tag{12}$$

indicating that the oscillators are decoupled and forced by a common oscillator with output $h(s)$.

From a variational equation analogous to Eq. (5), we have that all oscillators in Eq. (12) will be synchronized by the common forcing when

$$\alpha_1 < \sigma k_i^{1-\beta} < \alpha_2 \ \forall \ i. \tag{13}$$

For $\beta \neq 1$, it is enough to have a single node with degree very different from the others for this condition not to be satisfied for any σ. In this case, the complete synchronization becomes impossible because the corresponding oscillator cannot be synchronized. Within this approximation, the eigenratio is $R=(k_{\max}/k_{\min})^{1-\beta}$ for $\beta \leqslant 1$ and $R=(k_{\min}/k_{\max})^{1-\beta}$ for $\beta > 1$, where $k_{\min}=\min_i\{k_i\}$ and $k_{\max}=\max_i\{k_i\}$. The minimum of R is indeed achieved at $\beta=1$, in agreement with our numerical simulations.

Therefore, this simple mean-field approximation not only explains the results of Ref. [8] on the suppression of synchronizability due to heterogeneity in unweighted networks, but also predicts the correct condition for maximum synchronizability in weighted networks.

IV. MIXING RATE AND SYNCHRONIZABILITY

We relate the eingenratio R to the mixing rate of the process of diffusion introduced in Sec. III A and we argue that, for $\beta=1$ and large, sufficiently random networks, the synchronizability depends only on the mean degree of the network.

A. Mean-degree approximation

We now present a general physical theory for the eigenratio R. In what follows we focus on the case of maximum synchronizability ($\beta=1$). For $\beta=1$ and an arbitrary network, Eq. (8) can be written as

$$P = D^{-1/2}(I - H)D^{1/2}, \tag{14}$$

where $H=D^{-1/2}LD^{-1/2}$ as in Eq. (4). From the identity $\det(P-\lambda I)=\det(I-H-\lambda I)$, valid for any λ, it follows that the spectra of matrices P and H are related via $\rho(P)=1-\rho(H)$, and the uniform stationary state of the process $P^n y$ is associated with the null eigenvalue of the coupling matrix G. We then define the mixing rate as $\nu=\ln \mu^{-1}$, where

$$\mu = \lim_{n\mapsto\infty}\|P^n y - y_0\|^{1/n} \tag{15}$$

is the mixing parameter, $y_0=(1,1,\ldots,1)$ is the stationary distribution discussed in Sec. III B, y is an arbitrary initial distribution normalized as $\Sigma_i k_i y_i=\Sigma_i k_i$, and $\|\cdot\|$ is the usual Euclidean norm. In nonbipartite connected networks [20], we have $\lambda_N<2$ and the initial distribution y always converges to the stationary distribution y_0.

The convergence of the limit in Eq. (15) is dominated by the second largest eigenvalue of matrix P in absolute value, namely $\max_{i=2,\ldots N}|1-\lambda_i|$. (The largest eigenvalue is associated with the stationary state y_0.) As a result, for *any* network, the mixing parameter is

$$\mu = \max\{1 - \lambda_2, \lambda_N - 1\}. \tag{16}$$

Therefore, the mixing is faster in networks where the eigenvalues of the coupling matrix are concentrated close to 1.

The condition for the stability of the synchronized states also requires the eigenvalues of the coupling matrix to be close to 1, although through a slightly different relation ($R=\lambda_N/\lambda_2$ to be small). We can combine these two conditions to write an upper bound for the eigenratio R in terms of the mixing parameter:

$$R \leqslant \frac{1+\mu}{1-\mu}. \tag{17}$$

This relation is relevant because of its general validity and clear physical interpretation. We show that this upper bound is a very good approximation of the actual value of R in many networks of interest.

The mixing parameter μ can be expressed as an explicit function of the mean degree k. Based on results of Ref. [22] for random networks with given expected degrees, we get

$$\max\{1 - \lambda_2, \lambda_N - 1\} = [1 + o(1)]\frac{2}{\sqrt{k}}. \tag{18}$$

Moreover, the semicircle law holds and the spectrum of matrix P is symmetric around 1 for $k_{\min} \gg \sqrt{k}$ in the thermodynamical limit [22]. These results are rigorous for ensembles of networks with a given expected degree sequence and sufficiently large minimum degree $k_{\min}$, but our extensive numerical computation supports the hypothesis that the approximate relations

$$\lambda_2 \approx 1 - \frac{2}{\sqrt{k}}, \quad \lambda_N \approx 1 + \frac{2}{\sqrt{k}} \tag{19}$$

hold under much milder conditions. In particular, relations (19) are expected to hold true for any large, sufficiently random network with $k_{\min} \gg 1$. The rationale for this is that, for $\beta=1$, the diffusion *in* each node of one such network is the same as in a random homogeneous network with the same mean degree, where relations (19) are known to be satisfied [20].

Under the assumption that $1-\lambda_2 \approx \lambda_N - 1$, the eigenratio can be written as

$$R \approx \frac{1+\mu}{1-\mu}, \tag{20}$$

where μ is defined in Eq. (16). Therefore, the larger the mixing rate (smaller μ), the more synchronizable the network (smaller R) and vice versa. From Eq. (19), we have that the mixing parameter can be approximated as $\mu \approx 2/\sqrt{k}$ and the eigenratio can be approximated as

$$R \approx \frac{1 + 2/\sqrt{k}}{1 - 2/\sqrt{k}}. \tag{21}$$

Therefore, for $\beta=1$, the eigenratio R is primarily determined by the mean degree and does not depend on the number of oscillators and the details of the degree distribution.

This is a remarkable result because, regardless of the degree distribution, the network at $\beta=1$ is just as synchronizable as a random homogeneous network with the same mean degree, and random homogeneous networks appear to be asymptotically optimal in the sense that R approaches the absolute lower bound in the thermodynamical limit for large enough k [9].

B. Numerical verification

Now we test our predictions in models (i)–(iii) of SFN's, and we show that the synchronizability is significantly enhanced for $\beta=1$ as compared to the case of unweighted coupling ($\beta=0$).

As shown in Fig. 2, in unweighted SFN's, the eigenratio R increases with increasing heterogeneity of the degree distribution (see also Ref. [8]). But as shown in the same figure, the eigenratio does not increase with heterogeneity when the coupling is weighted at $\beta=1$. The difference is especially large for small scaling exponent γ, where the variance of the

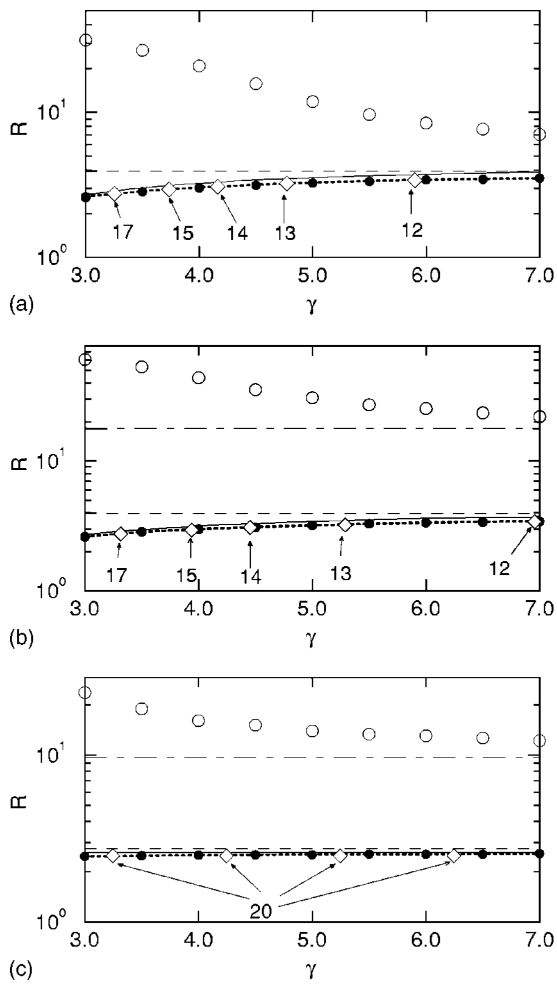

FIG. 2. Eigenratio R as a function of the scaling exponent γ: (a) random SFN's, (b) networks with expected scale-free sequence, and (c) growing SFN's, for $\beta=1$ (●) and $\beta=0$ (○). The other curves are the approximations of the eigenratio in Eqs. (20) (dotted lines) and (21) (solid lines), and the eigenratio for $\beta=1$ (dashed lines) and $\beta=0$ (dot-dashed lines) at $\gamma=\infty$. The ◇ symbols correspond to random homogeneous networks with the same mean degree of the corresponding SFN's (the degrees are indicated in the figure). The other network parameters are the same as in Fig. 1.

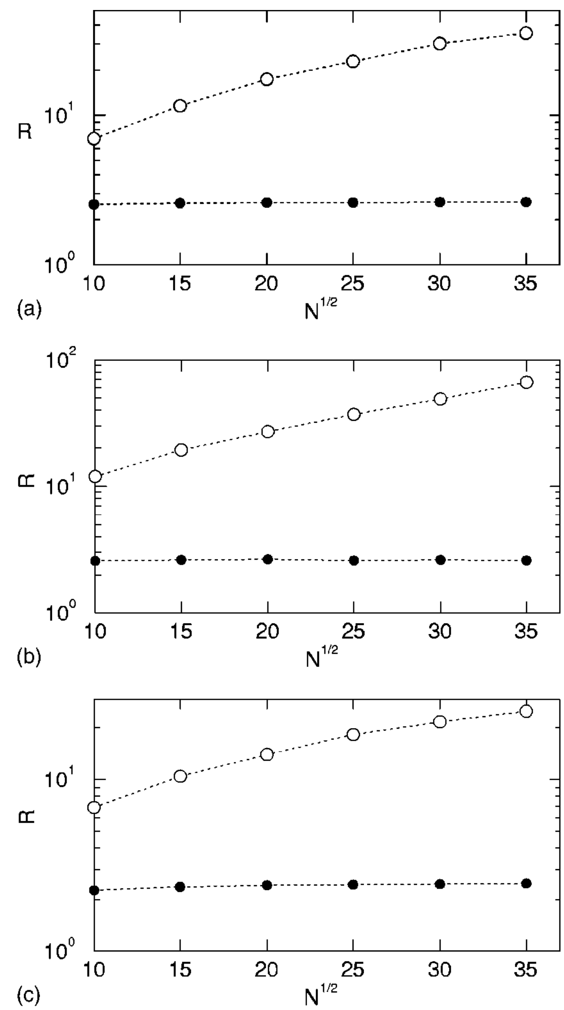

FIG. 3. Eigenratio R as a function of the number of oscillators for $\gamma=3$ and the SFN models in Figs. 2(a)–2(c), respectively. Dotted lines are guides for the eyes. The legend and other parameters are the same as in Fig. 2.

degree distribution is large and the network is highly heterogeneous (note that Fig. 2 is plotted in logarithmic scale). The network becomes more homogeneous as γ is increased. In the limit $\gamma=\infty$, random SFN's converge to random homogeneous networks with the same degree k_{min} for all the nodes [Fig. 2(a)], while networks with expected scale-free sequence converge to Erdős-Rényi random networks [31], which have links assigned with the same probability between each pair of nodes [Fig. 2(b)], and growing SFN's converge to growing random networks, which are growing networks with uniform random attachment [Fig. 2(c)]. As one can see from Figs. 2(b) and 2(c), the synchronizability is strongly enhanced even in the relatively homogeneous Erdős-Rényi and growing random networks; such an enhancement occurs also in SWN's.

For $\beta=1$, the eigenratio R is well approximated by the relations in Eq. (20) (Fig. 2, dotted lines) and Eq. (21) (Fig. 2, solid lines) for all three models of SFN's. This confirms our result that the synchronizability is strongly related to the mixing properties of the network [32]. For $\beta=1$, the eigenratio of the SFN's is also very well approximated by the eigenratio of random homogeneous networks with the same number of links [Fig. 2, ◇]. Therefore, for $\beta=1$, the variation of the eigenratio R with the heterogeneity of the degree distribution in SFN's is mainly due to the variation of the mean degree of the networks, which increases in both random SFN's and networks with expected scale-free sequence as the scaling exponent γ is reduced [Figs. 2(a) and 2(b)].

In Fig. 3, we show the eigenratio R as a function of the system size N. In unweighted SFN's, the eigenratio increases strongly as the number of oscillators is increased. Therefore, it may be very difficult or even impossible to synchronize large unweighted networks. However, for $\beta=1$, the eigenratio of large networks appears to be independent of the system size, as shown in Fig. 3 for models (i)–(iii) of SFN's. Similar results are observed in many other models of complex networks. All together, these provide strong evidence for our theory.

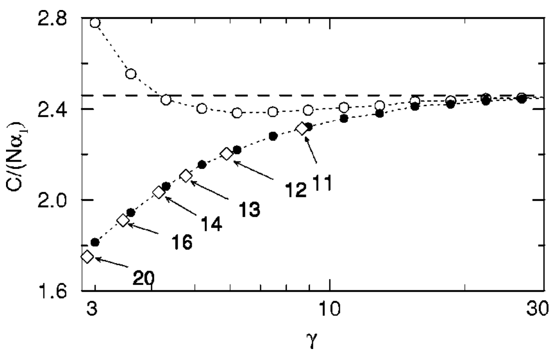

FIG. 4. Normalized cost $C/N\alpha_1$ as a function of the scaling exponent γ for random SFN's with $\beta=1$ (●) and $\beta=0$ (○), and for random homogeneous networks with the same mean degree (◇). The dashed line corresponds to $\gamma=\infty$. The other parameters are the same as in Fig. 1.

C. General bounds

We present bounds valid for *any* network weighted at $\beta=1$. In this case, if the network is connected but not globally connected and $N>2$, we have

$$1 + (N-1)^{-1} \leqslant R \leqslant 2NkD_{max}, \tag{22}$$

where k is the mean degree and D_{max} is the diameter of the network (maximum distance between nodes). This relation follows from the bounds $1/NkD_{max} \leqslant \lambda_2 \leqslant 1$ and $1+(N-1)^{-1} \leqslant \lambda_N \leqslant 2$ [20]. For being valid for any network regardless of its structure, the bounds in Eq. (22) are not tight for specific network models, such as random homogeneous networks. Nevertheless, they provide some insight into the problem. In particular, heterogeneity in the degree distribution is not disadvantageous in this case because it generally reduces the upper bound in Eq. (22), and this is a major difference from the case of unweighted networks considered previously [8].

V. COUPLING COST

Having shown that weighted networks exhibit improved synchronizability, we now turn to the problem of cost. We show that the total cost involved in the network of couplings is minimum at the point of maximum synchronizability ($\beta=1$).

The total cost C involved in the network of couplings is defined as the minimum (in the synchronization region) of the total strength of all directed links,

$$C = \sigma_{min}\sum_{i=1}^{N} k_i^{1-\beta}, \tag{23}$$

where $\sigma_{min}=\alpha_1/\lambda_2$ is the minimum coupling strength for the network to synchronize. We recall that α_1 is the point where the master stability function first becomes negative. For $\beta=1$, we have $C=N\alpha_1/\lambda_2$.

In heterogeneous networks, the cost at $\beta=1$ is significantly reduced as compared to the case of unweighted coupling ($\beta=0$), as shown in Fig. 4 for random SFN's. The

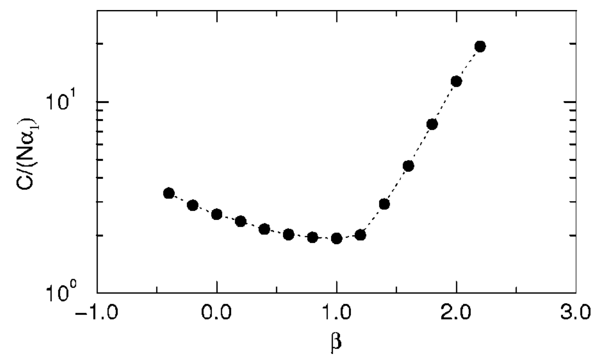

FIG. 5. Normalized cost $C/N\alpha_1$ as a function of β for random SFN's with scaling exponent $\gamma=3$. The other parameters are the same as in Fig. 1.

difference becomes more pronounced when the scaling exponent γ is reduced and the degree distribution becomes more heterogeneous. The cost for SFN's at $\beta=1$ is very well approximated by the cost for random homogeneous networks with the same mean degree (Fig. 4, ◇), in agreement with our analysis in Sec. IV A that, at $\beta=1$, the eigenvalue λ_2 is fairly independent of the degree distribution.

As a function of β, the cost has a broad minimum at $\beta=1$, as shown in Fig. 5 for random SFN's. A similar result is observed in other models of complex networks, including models (ii)–(iv) introduced in Sec. III B. This result is important because it shows that maximum synchronizability and minimum cost occur exactly at the same point. Therefore, cost reduction is another important advantage of suitably weighted networks.

VI. DIRECT SIMULATIONS

To confirm our analysis of enhanced synchronizability, we simulate the dynamics on networks of chaotic maps.

The example we consider consists of SFN's of logistic maps, $x_{n+1}=f(x_n)=ax_n(1-x_n)$, where the output function is taken to be $h(x)=f(x)$. In this case, the master stability function is negative for $(1-e^{-\Gamma_0})=\alpha_1<\alpha<\alpha_2=(1+e^{-\Gamma_0})$, where $\Gamma_0>0$ is the Lyapunov exponent of the isolated chaotic map. In the simulations of the dynamics, the maps are assigned to have random initial conditions close to the synchronization manifold.

We consider two values of the bifurcation parameter a for which the logistic map is chaotic: $a=3.58$, where $\alpha_2/\alpha_1\approx 19$, and $a=4.0$, where $\alpha_2/\alpha_1=3$. In both cases, our simulations show that, if $R<\alpha_2/\alpha_1$, then there is a finite interval of the overall coupling strength $\sigma_{min}<\sigma<\sigma_{max}$ where the network becomes completely synchronized after a transient time. Moreover, the simulations confirm that $\sigma_{min}=\alpha_1/\lambda_2$ and $\sigma_{max}=\alpha_2/\lambda_N$, as expected. In order to display the synchronization regions for different β in the same figure, we introduce $\sigma^*=\sigma\Sigma_i k_i^{1-\beta}$, $\sigma^*_{min}=\sigma_{min}\Sigma_i k_i^{1-\beta}$, and $\sigma^*_{max}=\sigma_{max}\Sigma_i k_i^{1-\beta}$.

In Fig. 6, we show the synchronization region $(\sigma^*_{min},\sigma^*_{max})$ as a function of the scaling exponent γ in random SFN's, for $\beta=1$ (●) and $\beta=0$ (○). The factor $\Sigma_i k_i^{1-\beta}$ is N for $\beta=1$ and

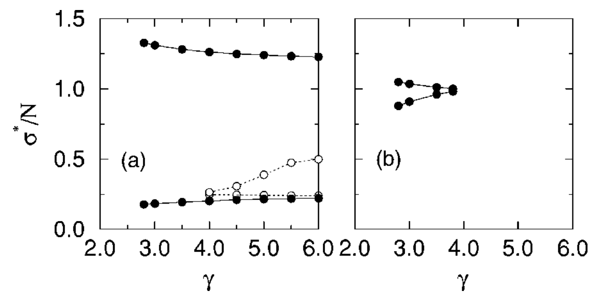

FIG. 6. Synchronization region $(\sigma^*_{min}, \sigma^*_{max})$ as a function of $\gamma > 2.8$ in random SFN's of logistic maps, for $\beta=1$ (●) and $\beta=0$ (○). The bifurcation parameter is (a) $a=3.58$ and (b) $a=4.0$. Averages are taken over 20 realizations of the networks. The other parameters are the same as in Fig. 1.

kN for $\beta=0$. For $a=3.58$, the networks are synchronizable for some σ^* in the region $\gamma \gtrsim 4.0$ if the couplings are unweighted ($\beta=0$) and in a wider region of γ if the couplings are weighted at $\beta=1$ [Fig. 6(a)]. In the region where both weighted and unweighted networks are synchronizable, the interval $(\sigma^*_{min}, \sigma^*_{max})$, in which synchronization is achieved, is much wider for $\beta=1$ than for $\beta=0$. In terms of the original coupling strength σ, the difference is even larger (k times larger). More strikingly, for $a=4.0$, unweighted networks do not synchronize for any coupling strength, but the networks weighted at $\beta=1$ do synchronize for $\gamma \lesssim 4.0$ in a nonzero interval of σ^* [Fig. 6(b)].

The cost C defined in Eq. (23) is exactly σ^*_{min}. In agreement with the results in Fig. 4, σ^*_{min} is clearly smaller for $\beta=1$ than for $\beta=0$ in the interval of γ where both weighted and unweighted networks are synchronizable [Fig. 6(a)]. All together, the results in Fig. 6 illustrate the enhancement of synchronizability and the reduction of cost in weighted networks.

VII. CONCLUDING REMARKS

Motivated by the problem of complex-network synchronization, we have introduced a model of directed networks with weighted couplings that incorporates the saturation of connection strength expected in highly connected nodes of realistic networks. In this model, the total strength of all *in*-links at a node i with degree k_i is proportional to $k_i^{1-\beta}$, where the parameter β is a measure of the degree-dependent saturation in the amount of information that a node receives from other nodes. In a network of oscillators, the weights $k_i^{1-\beta}$ can be alternatively interpreted as a property of the (*input* function of the) oscillators rather than a property of the links. We believe that this model can serve as a paradigm to address many problems of dynamics on complex networks.

Here we have studied complete synchronization of identical oscillators. We have shown that, for a given network topology, the synchronizability is maximum and the total cost involved in the network of couplings is minimum when $\beta=1$. For large, sufficiently random network with minimum degree $k_{min} \gg 1$, the synchronizability at $\beta=1$ is mainly determined by the mean degree and is fairly independent of the number of oscillators and the details of the degree distribution.

This should be contrasted with the case of unweighted coupling ($\beta=0$), where the synchronizability is strongly suppressed as the number of oscillators or heterogeneity of the degree distribution is increased. In the case $\beta=1$, the heterogeneity of the degree distribution is completely balanced and the networks are just as synchronizable as random homogeneous networks with the same mean degree.

Our results are naturally interpreted within a framework where the condition for the linear stability of synchronized states is related to the mixing rate of a diffusive process relevant for the communication between oscillators. Under mild conditions, we have shown that, the larger the mixing rate, the more synchronizable the network. In particular, in unweighted networks, the mixing rate decreases with increasing heterogeneity. This, along with the condition $\beta=1$ for enhanced synchronizability, explains and solves what we call the "paradox of heterogeneity." This paradox refers to the (apparently) paradoxal relation between the synchronizability and the average distance between oscillators, observed in heterogeneous unweighted networks [8], and is clarified when we observe that synchronizability is ultimately related to the mixing properties of the network.

We expect our results to be relevant for both network design and the understanding of dynamics in natural systems, such as neuronal networks, where the saturation of connection strength is expected to be important. Although we have focused mainly on SFN's, a class of networks that has received most attention, our analysis is general and applies to networks with arbitrary degree distribution.

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