Phase Synchronization of Chaotic Intermittent Oscillations

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We study phase synchronization effects of chaotic oscillators with a type-I intermittency behavior. The external and mutual locking of the average length of the laminar stage for coupled discrete and continuous in time systems is shown and the mechanism of this synchronization is explained. We demonstrate that this phenomenon can be described by using results of the parametric resonance theory and that this correspondence enables one to predict and derive all zones of synchronization.

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Synchronization of chaotic oscillations is a fundamental phenomenon observed in nature and science. Three main types of chaotic synchronization have been recently studied, namely, complete (or full) synchronization [1], generalized synchronization [2], and phase synchronization [3] (for reviews on chaotic synchronization, cf. [4]). Complete synchronization of identical systems occurs when the states of coupled systems coincide. Generalized synchronization implies that the output of one system is associated with a given function of the output of another system. Chaotic phase synchronization (CPS) is very similar to the synchronization of periodic oscillations and is manifested in the *coincidence of characteristic time scales* of coupled systems, but the amplitudes of oscillations often remain chaotic and practically uncorrelated. CPS has been detected in many natural, laboratory, and engineering systems (see [4]). Until now, CPS has been observed for rather phase-coherent chaotic attractors that occur after a cascade of period doubling bifurcations [3]. However, there are other typical routes to chaos, and the investigation of synchronization of the corresponding chaotic attractors is still absent.

In this Letter, we study phase synchronization of one such typical case: chaotic systems with a type-I intermittency. This type of intermittency has been observed in various experimental fields, such as lasers [5], fluid dynamics [6], oxidation processes [7], semiconductors [8], radially pulsating Tauri stars [9], plasma [10], or electronic circuits [11].

It is important to emphasize that chaotic intermittent motion does have its *characteristic time scale.* For type-I intermittency, a very large laminar stage with duration τ is changed by a very short turbulent stage (sometimes, just one jump) with duration T and then the next laminar stage begins. The average length of the laminar stage (ALLS) is defined as [12]

$$\langle \tau_0 \rangle \propto \frac{1}{\sqrt{\varepsilon - \varepsilon_{\rm cr}}}, \qquad (1)$$

where ε is a bifurcation parameter and $\varepsilon_{\rm cr}$ is the critical value when chaos sets in [13]. Moreover, one can introduce *a phase of the intermittent oscillations*, attributing to each interval between beginnings of the laminar stage a 2π phase increase:

$$\varphi = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \qquad t_n \le t < t_{n+1}, \qquad (2)$$

where t_n is the beginning time of the nth laminar stage.

The presence of the characteristic time scale allows an approach in terms of synchronization theory for two main problems: (i) synchronization by external periodic driving and (ii) mutual synchronization in a couple (an ensemble) of oscillators. In dynamical systems, which will be considered here, we expect the following synchronization behavior. In case (i), locking of a frequency, which corresponds to the characteristic time scale of the system with intermittency, takes place under sufficiently strong external driving. In case (ii), when two nonidentical systems are mutually coupled, their time scales should become equal if the coupling is strong enough, and this would manifest their mutual synchronization.

In order to show these effects, we first will treat analytically and numerically a quadratic 1D map that exhibits an intermittency type-I route to chaos which is subjected to external periodic driving. The locking of the ALLS will be shown to exist. The mechanism of synchronization will be explained as well. It will be shown that the considered synchronization can be described using results of the parametric resonance theory and that this correspondence enables one to predict and derive all zones of synchronization. Second, the existence of mutual synchronization between two coupled nonidentical quadratic 1D maps and, finally, external synchronization of a time-continuous system with type-I intermittency by external periodic driving will be demonstrated.

Let us start with a well-studied 1D map that exhibits the type-I intermittency route to chaos [14] under external periodic forcing:

$$x_{n+1} = f(x_n) + d\cos\omega n, \qquad (3)$$

where d and ω are the amplitude and the frequency of the external force, respectively , and $f(x)$ consists of the standard quadratic part and a somewhat arbitrary chosen return part that acts as a turbulent stage:

$$f(x) = \begin{cases} \varepsilon + x + x^2, & \text{if } x \leq 0.2 \\ g(x - 0.2) - \varepsilon - 0.24, & \text{if } x > 0.2. \end{cases} \quad (4)$$

Here g regulates the coherence properties of the chaotic attractor; in the case $g < 5$ the laminar stage duration is distributed in a rather narrow band, i.e., the chaotic behavior is highly coherent, but for $g > 5$ this distribution is rather broad. Both cases will be analyzed. First, we will focus on the case of a coherent chaotic attractor and let $g = 2$. We remind that without external force the map (3) demonstrates type-I intermittent behavior for $\varepsilon > 0$, i.e., $\varepsilon_{cr} = 0$.

Some aspects of the dynamics of intermittent behavior of similar models under external periodic driving [15–17] and mutually coupled identical systems [18] have been previously studied. However, the taken time scales of the intermittent motion and the driving signal were of substantially different order and synchronization has not been observed. In [18], complete chaotic synchronization was analyzed, which is out of the scope of this Letter.

We will focus on the synchronization phenomena occurring in (3). As we have mentioned before, ALLS $\langle \tau \rangle$ can be considered as the characteristic time scale in systems with intermittent behavior. We will calculate ALLS according to [14]. Using the fact that by chosen parameters the variable x changes slowly during the laminar stage, we rewrite the map in the form of a first-order time-continuous differential equation [14,15]:

$$\dot{x} = \varepsilon + x^2 + d\cos\omega t. \quad (5)$$

Using the change of the variable $x = -\dot{u}/u$, we obtain the Mathieu equation:

$$\ddot{u} + (\varepsilon + d\cos\omega t)u = 0. \quad (6)$$

In [15] this equation was studied by means of asymptotic methods under the assumption $\omega \gg \sqrt{\varepsilon}$, excluding any case of resonance. Quite the opposite, our interest will be strongly concentrated on the well-known cases of parametric resonance in (6).

The parametric resonance of the kth order can be achieved when the following relation is maintained [19]:

$$\sqrt{\varepsilon} \approx \frac{k}{2}\omega, \qquad k \in N. \quad (7)$$

When the conditions for the resonance are fulfilled, the solution is characterized by harmonic oscillations with the frequency $\frac{k}{2}\omega$ and exponentially growing amplitude $u = a\cos(\frac{k}{2}\omega t + \varphi)e^{p_k t}$, where a, φ are some constants, and p_k depends upon the number of the zone of the parametric resonance and the parameters of the systems. After transformation to the original variable, the parametric instability vanishes and one gets $x = \frac{k}{2}\omega\tan(\frac{k}{2}\omega t + \varphi) - p_k$, which yields the following for the ALLS in (3) in the synchronism:

$$\langle \tau_s \rangle \propto \frac{2}{k\omega}. \quad (8)$$

We take into account that proportionality coefficients in (1) and (8) defined by the return part of original systems [14] remain equal because they are not affected by the imposed weak external force. Then, we obtain

$$\langle \tau_s \rangle = \frac{2\sqrt{\varepsilon}\langle \tau_0 \rangle}{k\omega}, \quad (9)$$

where $\langle \tau_0 \rangle$ corresponds to the case of the autonomous map. So, inside a zone of parametric resonance that takes place in (6), the exponential growth does not affect the solution of (5), which is our concern, and all that matters is the frequency of the solution. On the other hand, being outside of a resonance zone, one gets a two-frequency solution in (6) and no synchronization exists in the original system (5): $u = ae^{i(k/2)\omega t}e^{i\Omega_k t} + \text{c.c.}$ For the first Mathieu zone ($k = 1$), the boundaries are given by

$$d = 4\varepsilon \left| \frac{\omega}{2\sqrt{\varepsilon}} - 1 \right|. \quad (10)$$

Outside the first zone of synchronization, the beating frequency is easy to write as

$$\Omega_1 = \frac{\omega}{4}\sqrt{\left(2 - \frac{\omega}{\sqrt{\varepsilon}}\right)^2 - \frac{d^2}{4\varepsilon^2}}, \quad (11)$$

which gives a quadratic scaling law, which is typical for phase synchronization on the border of the synchronization region.

Now we present our results of numerical simulations of the system (3) and compare them with our theoretical results. Throughout the Letter, we use the irrational frequency of external force $\omega = 0.001 \times 2\pi[(\sqrt{5} - 1)/2]$, i.e., proportional to the golden mean value, unless another value is specified. In Figs. 1(a) and 1(b), we show locking of the ALLS for different values of ε as the amplitude of

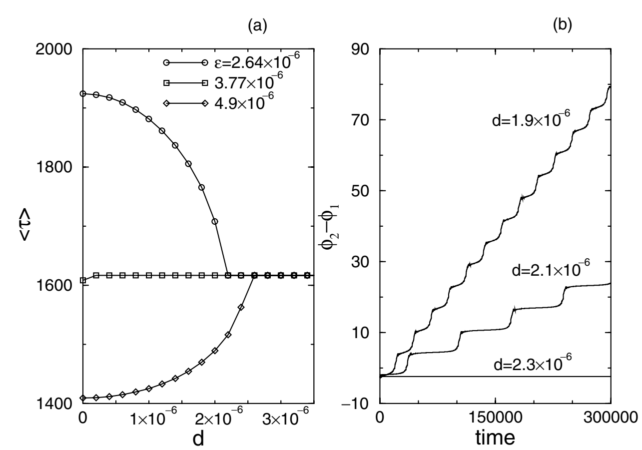

FIG. 1. (a) Locking of the ALLS by external periodic driving in (3). (b) Phase difference evolution in nonsynchronous ($d = 1.9 \times 10^{-6}$, 2.1×10^{-6}) and synchronous ($d = 2.3 \times 10^{-6}$) regimes in (3) for $\varepsilon = 2.64 \times 10^{-6}$.

the driving d is increased. It is easy to see that the theoretical result for the duration of the laminar stage of the synchronized motion (9) is quite well satisfied. Note that for the considered case $\langle\tau_s\rangle \approx \langle T_c^s\rangle = 2\pi/\omega$ [20]. Moreover, these relations also remain valid when the assumptions $d \ll \varepsilon$ and (7) do not hold. By approaching the synchronization plateau, the curves in Fig. 1(a) show a quadratic convergence, which confirms the quadratic scaling law predicted by (11). In addition, the phase locking is shown in Fig. 1(b) confirming the phase nature of the observed chaotic synchronization [the phase of the intermittent oscillations is defined according to (2)].

In Fig. 2(a), the first three zones of synchronization S_k, $k = 1, 2, 3$ and the region of absence of intermittency $I_{\rm off}$ are presented on the (d, ε) plane. Calculations show that the points of the synchronization regions that join the ε axis are positioned with accordance to Eq. (7). In Fig. 2(b), the regions S_1 and $I_{\rm off}$ are presented in more detail. The boundaries of the first zone of synchronization defined by relation (10) (shown by the ○-marked curve) give a remarkable coincidence with our numerical results. We find that the region of synchronization consists of two qualitatively different parts: In S_1^+ the Lyapunov exponent is positive, while in S_1^- it becomes negative, i.e., no chaos exists [Fig. 2(b)].

When the chaotic attractor in the autonomous map (3) ($d = 0$) is strongly noncoherent, synchronization is more difficult to achieve. Indeed, in this case the ALLS are broadly distributed in a long range of values, so *a priori* the possibility of adjusting the motion of this type may be problematic. Still, we carried out numerical simulations of (3) in the case $g = 11$. In accordance to [14], two typical time scales in the autonomous map (2) ($d = 0$) were observed. The first one is inherited from the case of $g < 5$, and presents itself a long-time laminar motion. The second one is a short-time laminar motion (about ten iterations of the map) which becomes more pronounced as g is increased. It turns out that synchronization of the long-time scale persists while, quite naturally, the short-time one is not synchronized. If one measures ALLS by taking into account only the long-time scale and neglecting fast passages, the synchronization by external driving is clearly observed [Fig. 3(a)]. It allows one to claim the existence of imperfect phase synchronization, if the driven system behaves non-phase-coherent [21].

We now demonstrate mutual synchronization of two coupled nonidentical quadratic 1D maps:

$$\begin{cases} x_{n+1} = f_1(x_n) + d(y_n - x_n) \\ y_{n+1} = f_2(y_n) + d(x_n - y_n), \end{cases} \qquad (12)$$

where $f_{1,2}(\cdot)$ are given by (4) with different values $\varepsilon_{1,2}$. In this system synchronization is manifested by the coincidence of the ALLS in the coupled maps. The transition to CPS and the time series in the synchronous regime are shown in Figs. 3(b) and 4, respectively.

Finally, we carried out numerical simulations of the Lorenz system (that exhibits type-I intermittency for $r \approx 166.06$ [12]) under multiplicative external driving (which may also be regarded as a modulation of the bifurcation parameter r [22]):

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz + dx\cos\omega t \\ \dot{z} = -bz + xy, \end{cases} \qquad (13)$$

where $b = -\frac{8}{3}$, $\sigma = 10$, and $\omega = 0.041\,77$. To test whether a laminar or a turbulent stage is observed, we calculated the sequences $\{y_n\}$ corresponding to intersections of the trajectory with the plane $\{x = 0, \dot{x} > 0\}$ and compared each value with the correspondent fixed point in the autonomous system on the edge of the tangent bifurcation. In computations of $\langle\tau\rangle$, a discrete time was used (one unit corresponded to the continuous-time

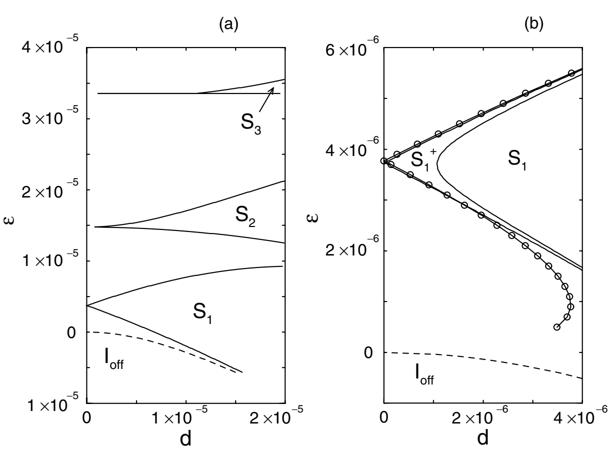

FIG. 2. (a) Three first zones of synchronization ($S_{1,2,3}$) and the region where intermittency is absent ($I_{\rm off}$). (b) The first zone of synchronization with positive (S_1^+) and negative (S_1^-) Lyapunov exponent. Theoretical border of the first synchronization zone (10) is the curve marked by ○.

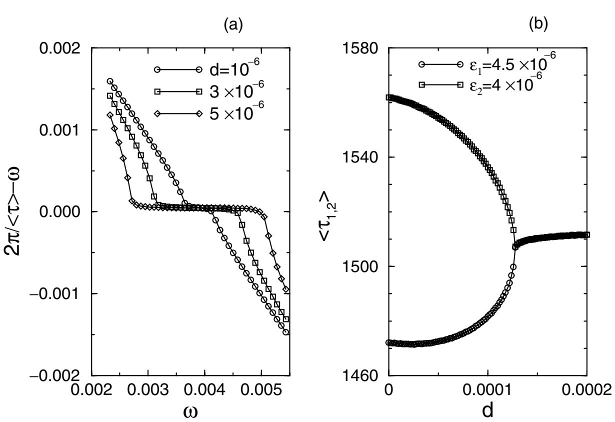

FIG. 3. (a) Locking of the ALLS by external periodic driving for a strongly non-phase-coherent chaotic attractor, $\varepsilon = 3.77 \times 10^{-6}$. (b) Mutual synchronization of two coupled nonidentical intermittent maps (12) for $g = 2$.

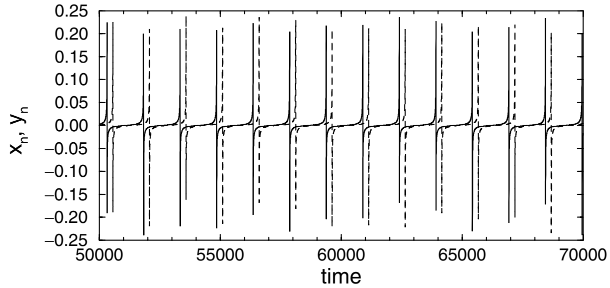

FIG. 4. Time series x_n (solid line) and y_n (dashed line) in a synchronous regime for $g = 2$, $\varepsilon_1 = 4.5 \times 10^{-6}$, $\varepsilon_2 = 4 \times 10^{-6}$, and $d = 0.000\,15$.

interval between intersections with the selected plane). In Fig. 5, we present results for different values of r (including subcritical ones). There exist plateaus of CPS and they are similar to those derived for the map. However, differences are also observed and resemble the imperfections of synchronization that we have discussed in the case of the strongly non-phase-coherent map (3). As the amplitude of the driving increased, synchronization gradually disappears (first comes a shallow slope, then a quadraticlike one). We have also observed CPS of mutually coupled Lorenz oscillators in the intermittent chaotic regime.

In conclusion, we have found the existence of CPS in systems with type-I intermittent behavior. The external and mutual locking of the ALLS for coupled discrete (1D map) and continuous (Lorenz oscillator) in time systems has been shown. We have demonstrated that the considered synchronization effects can be described using results of the parametric resonance theory and that this correspondence enables one to predict and derive all zones of synchronization. In addition, we emphasize another important impact of our study. As the investigations of low-dimensional (temporal) chaos are considered to be a basis for understanding high-dimensional (spatiotemporal) chaos, this problem may give a clue to synchronization and controlling of developed (spatiotemporal) turbulence that often looks similar to intermittent chaotic behavior. We also expect experimental studies on this finding in various fields, where type-I intermittency has been reported thus far (see [5–8,10,11]).

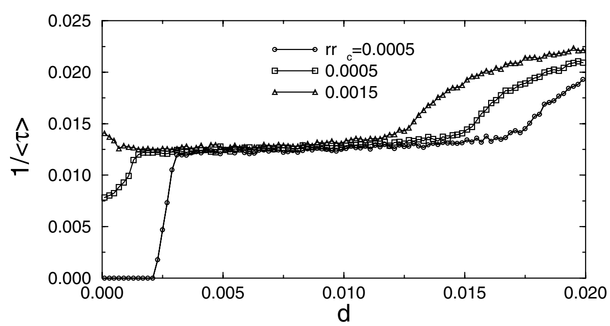

FIG. 5. Synchronization plateaus for the Lorenz system under external driving. Here $r_c = 166.061\,49$, $\frac{1}{\langle\tau\rangle} = 0$ corresponds to nonintermittent motion.

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