Measuring Weak Noise on Basis of Nonlinear Saturation of Prebifurcation Noise Amplification and the Noise-Dependent Hysteresis

E. Surovyatkina
Space Research Institute, Russian Academy of Sciences, Moscow, Russia < selena@iki.rssi.ru >
J. Kurths
Institut fur Physik, Potsdam Universitat, Am Neuen Palais 10, D-14469 Potsdam, German
<juergen@agnld.Uni-Potsdam.de>

The phenomenon of prebifurcation noise amplification is proposed as a promising indicator of bifurcations ("noisy precursor") in nonlinear processes in Geophysics. This phenomenon arises due to decreasing of damping coefficients just before bifurcation. A simple method for the estimation of the forced fluctuation variance is suggested which is based on results of linear theory up to the boundary of its validity. The upper level for the fluctuation variance before the onset of the bifurcation is estimated from the condition that the contribution of the nonlinear term becomes comparable (in the sense of mean squares) with that of the linear term. The method has proved to be efficient for two simple bifurcation models (period doubling bifurcation and bifurcation of spontaneous symmetry breaking) and might be helpful in application to geophysics problems. The transition of a nonlinear system through the bifurcation point offers a new possibility for estimating the internal noise using the magnitude of the noise-dependent hysteresis curve, which occurs when the critical point is shifted in the forward and backward direction.

1. Introduction

As is known, [1, 2]. ____

Alternative mechanism of ____________might be connected with *prebifurcation noise amplification* in areas subjected to bifurcation. The main goal of this publication is to attract the attention of geophysicists to this phenomenon [3, 4] and thereby to stimulate searches for specific mechanisms that might be responsible for noise amplification in geophysic.

The phenomenon of prebifurcation noise amplification is shortly outlined below by the examples of period doubling bifurcation in nonlinear map (Sect.2) and spontaneous symmetry breaking bifurcation in nonlinear oscillator (Sect.3) with a special emphasis on the transition from linear regime [3] to the regime of nonlinear saturation of amplification [4,5].

2. Prebifurcation noise amplification at the threshold of period doubling bifurcation

Nonlinear noisy map

$$x_{n+1} = F(x_n, \mu) + f_n , \tag{1}$$

where μ is a control parameter, and f_n is an external δ-correlated random process, $\langle f_n f_m \rangle = \sigma_f^2 \delta_{nm}$, has a stable point $\bar{x}$ which obeys the equation $\bar{x} = F(\bar{x}, \mu)$. Fluctuations $\xi_n = x_n - \bar{x}$ near stable state are governed by the equation

$$\xi_{n+1} = \gamma \xi_n + \varepsilon \xi_n^2 + \ldots + f_n , \tag{2}$$

where $\gamma = \dfrac{dF(\bar{x})}{dx}$ and $\varepsilon = \dfrac{1}{2}\dfrac{d^2 F(\bar{x})}{dx^2}$. Bifurcation of period doubling arises when the modulus of multiplier γ exceeds a unit: $|\gamma| > 1$.

In frame of linear approach, when Eq. (2) takes the form $\xi_{n+1} = -\gamma\xi_n + f_n$, variance of fluctuations $\langle \xi_n^2 \rangle \equiv \sigma_f^2$ is given by

$$\left(\sigma_\xi^2\right)_{lin} = \frac{\sigma_f^2}{1-\gamma^2} \cong \frac{\sigma_f^2}{2\alpha}, \tag{3}$$

where parameter $\alpha = 1-|\gamma|$ characterizes closeness of the nonlinear system to bifurcation threshold $|\gamma|$=1. According to Eq.(3), in frame of linear approach fluctuation variance σ_ξ^2 tends to the infinity at $\alpha \to 0$, what corresponds to the case studied by Wiesenfeld [3].

Linear theory is valid until the linear term in Eq. 2 becomes comparable (in the statistical sense) with the quadratic one, that is when

$$(1-\gamma^2)\sigma_\xi^2 \approx \varepsilon^2 \langle \xi^4 \rangle. \tag{4}$$

Using relation $\langle \xi_n^4 \rangle \approx 3\sigma_\xi^2$, as for Gaussian values, one can estimate (from Eqs. (3) and (4)) the minimal value $\alpha_{\min}$ limiting the area of linear theory validity: $\alpha_{\min} = \sqrt{3}\varepsilon\sigma_f / 2$.
Substitution of $\alpha_{\min}$ into Eq. (3) allows estimating the level of saturated fluctuation [4]:

$$(\sigma_\xi^2)_{nonlin} \approx \frac{\sigma_f}{\sqrt{3}\varepsilon}. \tag{5}$$

The described dependence of σ_ξ^2 on α is presented on Fig. 1: at $\alpha > \alpha_{\min}$, one can use the results of linear theory, Eq. (3) (unlimited dashed line), whereas at $\alpha < \alpha_{\min}$, nonlinear estimate (5) enters into the play (horizontal pointed line).

The estimates presented above, are in good agreement with the results of numerical simulation, performed in [4] and shown by a continuous line at Fig. 1. Estimate (5) happens to be only 30-40% less as compared to numerical data.

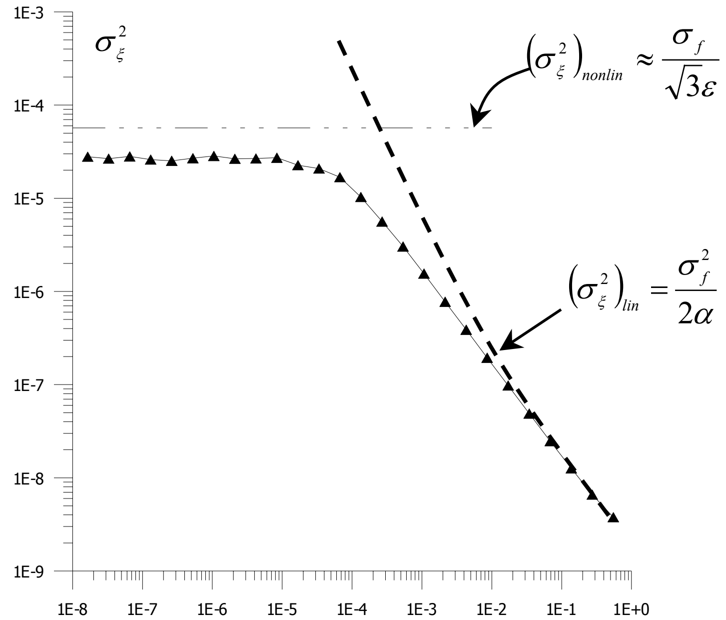

Fig. 1. Prebifurcation noise amplification for period doubling bifurcation in quadratic map $F(x) = \mu - x^2$ (σ_f^2=3.3·10^{-9}, ε=1).

3. Prebifurcation noise amplification in nonlinear oscillator experiencing bifurcation of spontaneous symmetry breaking

Nonlinear oscillator, described by equation

$$\ddot{\xi} + 2\beta\dot{\xi} + B\xi + A\xi^3 = f(t), \quad (6)$$

where β is damping coefficient, admits bifurcation of spontaneous symmetry breaking at B_c=0: unique stable state $\overline{\xi} = 0$ at $B > 0$ converts into pair of stable states $\overline{\xi} = \pm\sqrt{-\frac{B}{2A}}$ at $B < 0$.

In the linear approximation (nonlinear term ξ^3 is considered to be negligibly small) variance $\sigma_\xi^2 = \langle \xi^2 \rangle$ is given by

$$(\sigma_\xi^2)_{lin} = \frac{\sigma_f^2 \tau_f}{2\beta B}, \quad (7)$$

where τ_f is correlation time of the process $f(t)$. According to Eq. (7), in frame of linear theory fluctuation variance σ_ξ^2 tends to infinity, when B is approaching to the bifurcation point $B_c = 0$. The results of linear theory are shown at Fig.2 by a dashed curve.

Variance of the elastic force $B\xi + A\xi^3$ is equal to

$$Var(B\xi + A\xi^3) = B^2\langle \xi^2 \rangle + 2AB\langle \xi^4 \rangle + A^2\langle \xi^6 \rangle. \quad (8)$$

The contribution of nonlinear term $2AB\langle \xi^4 \rangle \approx 6AB\sigma_\xi^4$ in Eq. 8 ($\langle \xi^4 \rangle$ is considered to be $3\sigma_\xi^4$, as for Gaussian process) becomes comparable with the contribution of linear term $B^2\sigma_\xi^2$ when

$$\sigma_\xi^2 \approx \frac{B}{6A}. \quad (9)$$

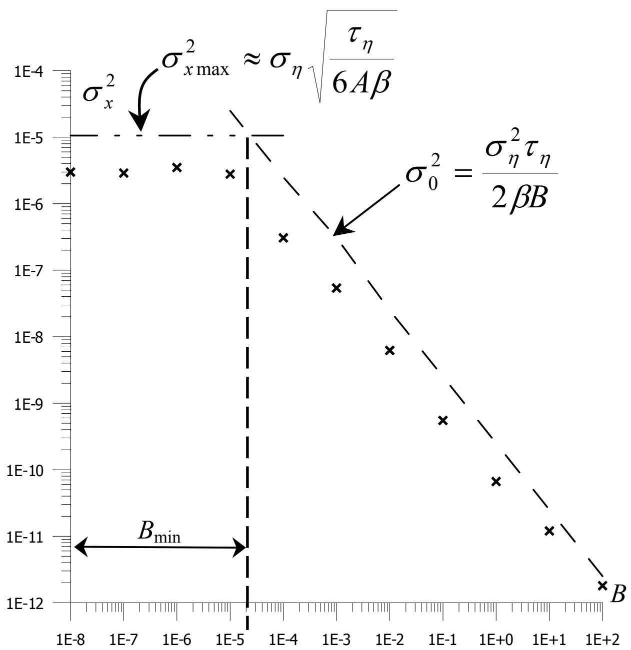

Fig. 2. Prebifurcation noise amplification for bifurcation of spontaneous symmetry breaking (σ_η^2=10^{-8}, τ_η=$3.5 \cdot 10^{-3}$, β=0.1, A=0.5)

Taking for σ_ξ^2 linear expression (7), one can estimate the minimal value of $B_{\min}$, restricting the area of applicability of linear theory:

$$B_{\min} \approx \sigma_f \sqrt{\frac{3A\tau_f}{\beta}} \,. \tag{10}$$

This value, substituted in Eq. (9) provides nonlinear estimate for fluctuation saturation at bifurcation threshold $B = 0$:

$$\left(\sigma_\xi^2\right)_{nonlin} \approx \sigma_f \sqrt{\frac{\tau_f}{12A\beta}} \,. \tag{11}$$

This estimate is presented by a horizontal dotted line in Fig.2.

4. On Measuring Weak Noise

A noticeable increase in the variation of fluctuations σ_ξ^2 near the bifurcation threshold can be used as the basis of a method for measuring weak noise in the system under study. In contrast to the highly unstable pregeneration noise, which appears due to sporadic transitions of the system into the generation mode, the prebifurcation noise is more stable and ensures more reliable measurements. This is true not only for period-doubling bifurcations, but also for many other bifurcation types that do not involve a large increase in the oscillations when transiting through the critical point as is the case for Landau - Hopf bifurcations. Our approach is based on a comparison of the maximal variation $(\sigma_\xi^2)_{\max} \sim \sigma_f$ at the bifurcation point with the variation $\sigma_\xi^2 \approx \sigma_f^2$ away from that point. An estimate of the noise variation σ_f^2 on the basis of the mean shift $\langle \xi \rangle$, which appears due to a sort of detection of fluctuations on the nonlinearity of the system, seems less reliable. The possibility of measuring weak noise by measuring $(\sigma_\xi^2)_{\max}$ is limited on the duration of measurements. According to [3], the duration of the sample must be not less than the transient time $n_{\text{trans}} \sim 1/\sigma_f$.

5. Noise-Dependent Hysteresis

6. Use of hysteresis in bifurcation systems for noise measurement

Conclusion

The most characteristic feature of the saturation regime is that fluctuation variance σ_ξ^2 is proportional to noise standard deviation σ_f [Eq. (5) and Eq. 11], unlike linear regime, when $\sigma_\xi^2 \approx \sigma_f^2$ [Eq. (3) and Eq. (7)].

Regardless specific bifurcation mechanism, prebifurcation noise amplification in respect to geophysical processes might be of practical interest at least from two points of view: as additional mechanism for ___________________________and as indicator of forthcoming bifurcations ("noisy precursor").

On the basis of the phenomenon under study, a method for measuring the variance of weak noise in nonlinear systems is proposed. The applicability of this method is limited by the necessity to perform rather long-term observations.

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References

[1]
[2]
[3] K. Wiesenfeld, J. Stat. Phys. **38** (1985) 1071.
[4] Yu.A. Kravtsov, S.G. Bilchinskaya, O.Ya. Butkovskii, I.A. Rychka, and E.D. Surovyatkina, JETP **93** (6) (2001) 1323.
[5] Yu. A. Kravtsov, E. D. Surovyatkina, Physics Letters A, **319** (3-4), pp. 348-351 (2003)
[6] Baesens C. Physica-D. 1991. V.53. N2-4. P.319-376.
[7] O. Ya. Butkovskii, Yu. A. Kravtsov, E. D. Surovyatkina, Zh. Tekh. Fiz., **67**(9), 128 (1997) [Tech.Phys.42, 1099 (1997)].