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Noise-enhanced synchronization of homoclinic chaos in a CO_2 laser

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Many chaotic oscillators have rather coherent phase dynamics but strong fluctuation in the amplitudes. Conversely, homoclinic chaos is characterized by quite regular spikes but strong fluctuation in their time intervals. We study the effects of noise on the synchronization of homoclinic chaos to a weak periodic signal and demonstrate numerically and experimentally in a CO_2 laser system that noise enhances synchronization of homoclinic chaos. The system exhibits both conventional resonance versus driving frequency and stochastic resonance with respect to noise intensity.

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Resonant response of a nonlinear system to a weak driving signal has been investigated in various contexts. In a self-sustained periodic oscillator, the system adjusts its time scale, achieving frequency and phase locking to the driving signal. This phenomenon of conventional resonance, characterized by an Arnold tongue synchronization region is of fundamental importance [1]. Recently, the study of phase synchronization (PS) has been extended to chaotic model oscillators [2,3] and to several experiments, such as lasers [4]. For example, in the chaotic Rössler oscillator, a phase variable can be defined which is associated with the time scales of the oscillations, e.g., the return time T between two successive crossing of a Poincaré section [2]. This system displays very coherent phase dynamics due to a small fluctuation of T, although the amplitudes fluctuate strongly. This property is quite general in chaotic oscillations resulting from a period-doubling bifurcation [5], and PS and conventional resonance occur similar to coupled periodic oscillators [2,3].

Noise usually has a destructive effect on PS by inducing phase slips and shrinking the synchronization region [6,7]. On the other hand, noise may play a constructive role in enhancing the response through stochastic resonance (SR) [8]. Stochastic resonance has also been studied from the viewpoint of noise-enhanced synchronization of the switching events to the external signal, because noise controls the average switching rate of the system and the response is optimal when it is close to that of the external signal [9–11]. This resonance behavior, however, is not the same as conventional resonance in coupled self-sustained oscillators: while the synchronization exhibits a resonancelike behavior with the change of noise intensity at a fixed driving frequency, it does not display resonancelike behavior as a function of the driving frequency [11]. When the driving signal fluctuations are much slower than all system time scales, SR is independent of signal frequencies; while it shows a sensitivity to higher signal frequencies in excitable systems [12,13]. In phase-coherent chaotic oscillators, noise may also play a constructive role to induce [14] or enhance PS in the weak coupling regime [15].

Homoclinic chaos [16] represents a class of chaotic oscillations that exhibit quite different behavior as compared to phase-coherent chaotic oscillations. Typically, these chaotic oscillators possess a saddle point S embedded in the chaotic attractor, with an unstable manifold weaker than the stable one (the Shilnikov condition for homoclinic chaos [16]: the eigenvalues $|\lambda_u|<|\lambda_s|$). The chaotic trajectories leaving a neighborhood of S along its unstable manifold have very close recurrence to S along its stable manifold. Typical dynamics is characterized by rather regular orbits in the phase space and widely fluctuating time intervals T between successive returns (Fig. 1), because the trajectory slows down considerably and T depends on how close the orbit approaches S. Such a structure underlies spiking behavior in many neuron [17], chemical [18], laser [19], and El Niño [20] systems. Noise acts in homoclinic chaotic systems in a quite different way [20]. The motion is sensitive to noise along the weaker unstable manifold, which on average makes the trajectory leave the neighborhood of S earlier and reduces the average interval T. So far, such effects of noise on PS of homoclinic chaos have not been addressed.

Here we show that a small noise changes not only the average value $T_0=\langle T\rangle_t$, but also reduces the fluctuations of T. As a result, noise *enhances* PS and the system displays both conventional and stochastic resonances.

We demonstrate these nontrivial effects of noise in a single-mode CO_2 laser, both numerically and experimentally. The experimental setup consists of a CO_2 laser with an intracavity loss modulator, driven by a feedback signal which is proportional to the laser output intensity. The system is operating in a homoclinic chaos regime where the laser output consists of a chaotic sequence of spikes [19,21] (Fig. 1). The pump parameter p_0 is modulated by an external periodic signal with amplitude A and frequency f_e,

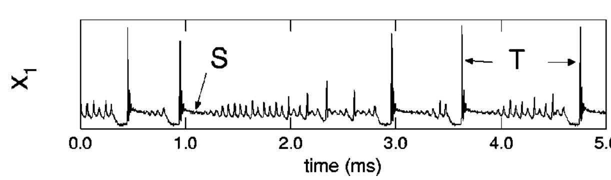

FIG. 1. Time series of the laser output intensity in the experimental CO_2 system without external signal and noise.

$$p(t)=p_0[1+A\sin(2\pi f_e t)]. \tag{1}$$

To investigate the role of external noise, a Gaussian noise generator is inserted into the feedback loop. The noise generator has a high frequency cutoff at 50 kHz, which can be regarded as a white noise source.

We first carry out numerical simulations on the model,

$$\dot{x}_1=k_0x_1(x_2-1-k_1\sin^2 x_6), \tag{2}$$

$$\dot{x}_2=-\gamma_1x_2-2k_0x_1x_2+gx_3+x_4+p(t), \tag{3}$$

$$\dot{x}_3=-\gamma_1x_3+gx_2+x_5+p(t), \tag{4}$$

$$\dot{x}_4=-\gamma_2x_4+zx_2+gx_5+zp(t), \tag{5}$$

$$\dot{x}_5=-\gamma_2x_5+zx_3+gx_4+zp(t), \tag{6}$$

$$\dot{x}_6=-\beta\left(x_6-b_0+\frac{rx_1}{1+\alpha x_1}\right)+D\xi(t), \tag{7}$$

which describes accurately the experimental system [21]. Here, x_1 represents the laser output intensity, x_2 the population inversion between the two resonant levels, x_6 the feedback voltage signal which controls the cavity losses, while x_3, x_4, and x_5 account for molecular exchanges between the two levels resonant with the radiation field and the other rotational levels of the same vibrational band. Furthermore, k_0 is the unperturbed cavity loss parameter, k_1 determines the modulation strength, g is a coupling constant, γ_1,γ_2 are population relaxation rates, p_0 is the pump parameter, z accounts for an effective number of rotational levels, and β, b_0, r, α are, respectively, the bandwidth, the bias voltage, the amplification, and the saturation factors of the feedback loop. With the following parameters $k_0=28.5714$, $k_1=4.5556$, $\gamma_1=10.0643$, $\gamma_2=1.0643$, $g=0.05$, $p_0=0.016$, $z=10$, $\beta=0.4286$, $\alpha=32.8767$, $r=160$, and $b_0=0.1032$, the model reproduces the regime of homoclinic chaos observed experimentally [21]. The previous study [21] did not take into account the intrinsic noise present in the experimental system. We have measured the noise in the feedback variable (x_6) in the case when the laser is off. This enables us to estimate an intrinsic noise intensity $D\approx 7$ mV, which is about 0.14% of the feedback signal x_6 in the experimental system. In the model, $D=0.0005$ is equivalent to the intrinsic noise intensity in x_6.

Without noise and driving signal, the orbit approaches S via a few quickly decaying oscillations (stable manifold) and leaves S via a series of slowly growing ones (unstable manifold). It may have different number of oscillations before generating a large spike, depending on the distance from S at the previous reinjection. As a result, the model displays a broad range of time scales, and there are many peaks in the distribution $P(T)$ of the interspike interval T [Fig. 2(a)]. With a small noise ($D=0.0005$), the orbits can no longer perform some oscillations very close to S, resulting in a clear change in the time scales: $P(T)$ is now characterized by a dominant peak followed by a few exponentially decaying ones [Fig. 2(c)]. This distribution of T is typical for small D in the range $D=0.000\,05$–0.002. The experimental system with only intrinsic noise (equivalent to $D=0.0005$ in the model) has a very similar distribution $P(T)$ (not shown). At larger intensity $D=0.01$, noise eliminates most of the oscillations around S; the fine structure of the peaks is smeared out and $P(T)$ becomes a unimodal peak with a lower height [Fig. 2(e)]. Note that the average value $T_0(D)$ of T decreases with increasing D. The measure of the coherence [22] of the spike trains by $R=T_0(D)/\sigma_T$, where σ_T is the standard deviation of $P(T)$, shows a maximal value at $D\approx 0.013$. Thus the spiking sequence displays a coherence resonance feature similar to excitable systems [22].

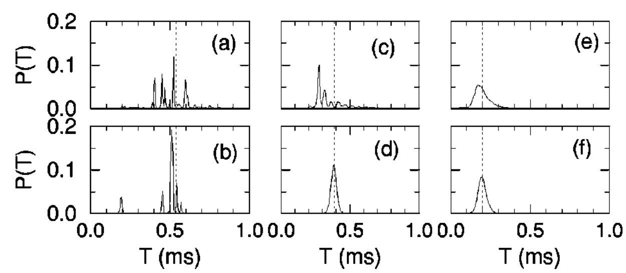

FIG. 2. Probability density of interspike intervals of noise-free [$D=0$, (a),(b)] and noisy [$D=0.0005$, (c),(d); $D=0.01$, (e),(f)] laser model. Upper panel, without external forcing: $A=0$; lower panel, with forcing $A=0.01$. The signal period T_e in (b), (d), and (f) corresponds to the average interspike interval $T_0(D)$ (vertical dotted lines) of the unforced model in (a), (c), and (e), respectively.

As a result of noise-induced changes in time scales, the model displays quite different response to a weak signal ($A=0.01$) with a frequency $f_e=f_0(D)=1/T_0(D)$, i.e., equal to the average spiking rate of the unforced model. At $D=0$, $P(T)$ of the forced model still has many peaks [Fig. 2(b)], while at $D=0.0005$, T is sharply distributed around the signal period $T_e=T_0(D)$ [Fig. 2(d)]. However, at larger intensity $D=0.01$, $P(T)$ becomes lower and broader again [Fig. 2(f)]. To examine phase synchronization due to the driving signal, we compute the phase difference $\theta(t)=\phi(t)-2\pi f_e t$. Here the phase $\phi(t)$ of the laser spike sequence is simply defined as [2] $\phi(t)=2\pi[k+(t-\tau_k)/(\tau_{k+1}-\tau_k)],(\tau_k<t\leq\tau_{k+1})$, where τ_k is the spiking time of the kth spike. As seen in Fig. 3, at $D=0$, the phase of the laser model is not locked by the external forcing. On the contrary, with a small noise $D=0.0005$, phase slips occur very rarely and phase locking becomes almost perfect when noise generates a characteristic time scale in the system. At stronger intensity $D=0.01$, noise becomes dominant over the signal

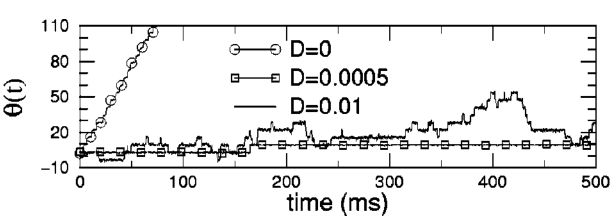

FIG. 3. Phase difference between the laser model and the driving signal (as in Fig. 2) at various noise intensities.

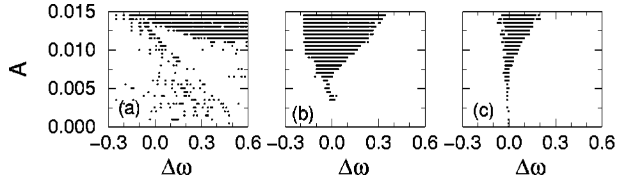

FIG. 4. Synchronization region of the laser model at various noise intensities. A dot is plotted when $|\Delta\Omega|\leqslant 0.003$. (a) $D=0$, (b) $D=0.0005$, and (c) $D=0.005$.

around the saddle S, and it induces many randomlike phase slips. The behavior is similar for driving frequencies close to $f_0(D)$.

We have investigated the synchronization region (1:1 response) of the laser model in the parameter space of the driving amplitude A and the relative initial frequency difference $\Delta\omega=[f_e-f_0(D)]/f_0(D)$, where the average frequency $f_0(D)$ of the unforced laser model is an increasing function of D. The actual relative frequency difference in the presence of the signal is calculated as $\Delta\Omega=(f-f_e)/f_0(D)$, where $f=1/\langle T\rangle_t$ is the average spiking frequency of the forced laser model. The synchronization behavior of the noise-free model is quite complicated and featureless [Fig. 4(a)]: at weak amplitudes (about $A<0.012$), there does not exist a tonguelike region similar to the Arnold tongue in phase-coherent oscillators; for a fixed A, $\Delta\Omega$ is not a monotonous function of $\Delta\omega$ and it vanishes only at some specific signal frequencies [also see Fig. 5(a), $D=0$]; at stronger driving amplitudes (about $A>0.012$), the system becomes periodic at a large frequency range. The addition of a small noise, $D=0.0005$, drastically changes the response: a tonguelike region [Fig. 4(b)], where effective frequency locking ($|\Delta\Omega|\leqslant 0.003$) occurs, can be observed similar to that in usual noisy phase-coherent oscillators. The synchronization region shrinks at a stronger noise intensity $D=0.005$ [Fig. 3(c)].

The very complicated and unusual response to a weak driving signal in the noise-free model has not been observed in the experimental system due to the intrinsic noise whose intensity is equivalent to $D=0.0005$ in the model. As has been reported recently [23], the experimental system without an additional external noise displays similar tonguelike synchronization region as in Fig. 4(b). The experimental obser-

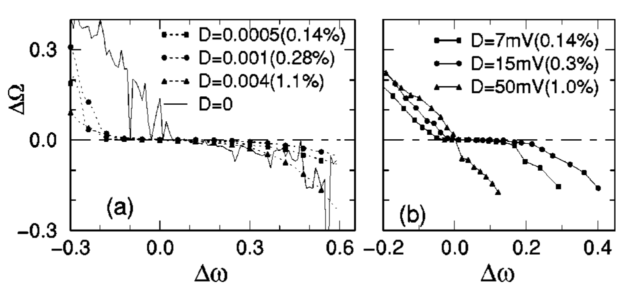

FIG. 5. Noise-enhanced PS: a comparison between model and experimental systems. (a) Model, $A=0.01$. (b) Experiment: signal amplitude 10 mV ($A=0.01$); the noise intensity denotes total noise measured in the feedback loop, and $D=7$ mV corresponds to the intrinsic noise. In both cases, the noise intensities are also indicated in percent of the feedback signal x_6.

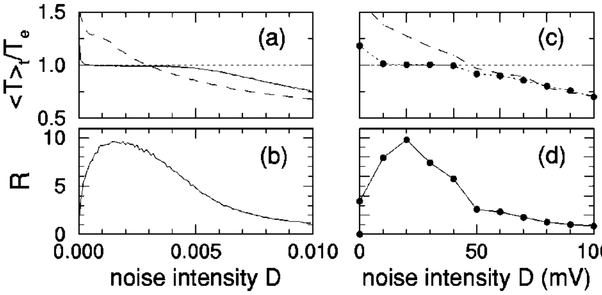

FIG. 6. Stochastic resonance for a fixed driving period. Left panel: model, $A=0.01, T_e=0.3$ ms. Right panel, experiments: forcing amplitude 10 mV ($A=0.01$) and period $T_e=1.12$ ms; here the noise intensity D is of the added external noise. Upper panel: noise-induced coincidence of average time scales (dashed line, $A=0$) and synchronization region. Lower panel: coherence of the laser output. $\alpha=0.1$ in Eq. (8).

vations of the noise-induced changes in the time scales and response to an external driving signal are consistent with the model. A comparison of numerical and experimental results for a similar noise range is shown in Fig. 5. Very importantly, synchronization in the experimental system has been enhanced further by adding some external noise, especially for $\Delta\omega>0$; an external noise too strong degrades synchronization again, as seen in Fig. 5(b).

Thus, noise can play a constructive role to enhance frequency locking and PS of homoclinic chaos to a weak driving signal. Without noise, the model system exhibits a very complicated response to the signal due to a broad distribution of time scales; whereas a small noise eliminates some of the small oscillations close to S and generates a dominant time scale, and the system displays a locking with respect to the signal frequency, as conventional resonance in phase-coherent oscillators.

Furthermore, the PS behavior is optimized at a certain noise intensity, similar to SR [8–11]. We study how this SR behavior is affected by noise intensity. In bistable or excitable systems, SR occurs when the noise-controlled average time scale is close to that of the driving signal [9–12]. Here, in the unforced homoclinic chaotic lasers the average interspike interval $T_0(D)$ decreases with increasing noise intensity. SR can be observed for a fixed signal period T_e. We have employed the following measure of coherence as indicator of stochastic resonance [13]

$$R=\frac{T_e}{\sigma_T}\int_{(1-\alpha)T_e}^{(1+\alpha)T_e}P(T)dT, \tag{8}$$

where $0<\alpha<0.25$ is a free parameter. As pointed out in Ref. [13], this indicator takes into account both the fraction of spikes with an interval roughly equal to the forcing period T_e and the jitter between spikes. SR of the 1:1 response to the driving signal with a fixed period has been demonstrated both in the model and in the experimental system by the ratio $\langle T\rangle_t/T_e$ and R (Fig. 6). For $T_e<T_0(0)$, there exists a synchronization region where $\langle T\rangle_t/T_e\approx 1$. The noise intensity optimizing the coherence R is smaller than that inducing coincidence of $T_0(D)$ and T_e [dashed lines in Fig. 6(a),(c)]. It

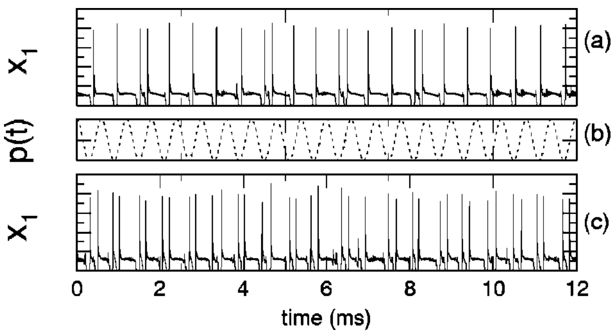

FIG. 7. Noise-enhanced 2:1 response of the laser model. $A=0.01, T_e=0.6$. (a) Laser output x_1 at $D=0$, (b) external signal, and (c) x_1 at $D=0.004$.

turns out that maximal coherence occurs when the dominant peak of $P(T)$ is located at T_e. For $T_e > T_0(0)$, noise may induce an $n:1$ response where the laser produces *n* spikes per signal period. For example, at $T_e=0.6$, a 2:1 response can be observed in the laser model which generates two spikes with alternately small and large intervals T_1 and T_2 satisfying $T_1+T_2=T_e$, as seen in Fig. 7. The $n:1$ response also exhibits a locking and resonance with the change of both the signal frequency and noise intensity. This different noise-induced synchronization has not been reported in usual SR systems. Conversely, in usual SR systems, at large T_e numerous randomlike firings per period cause an exponential background of $P(T)$, and at small T_e a $1:n$ response may occur, which means an aperiodic firing sequence with one spike for *n* driving periods on average [8,12,13]; in both cases, the sequences are irregular.

Note that the response of the homoclinic chaos to noise, i.e., more regular spike intervals with a smaller mean value, is similar to excitable systems where resonances with respect to both signal frequency and noise intensity can also be observed [12,24]. However, a noise-induced phase locking with respect to the signal frequency, especially for the $n:1$ ratios, to our knowledge, has not been demonstrated in excitable systems for rather weak signal.

In summary, we have shown that in homoclinic chaotic systems which are characterized by a strong fluctuation of the interspike interval, the time scales become more regular in the presence of a small noise. Consequently, the PS of the system to a weak driving signal can be enhanced significantly, and the noisy system exhibits locking and resonance with the change of both the signal frequency and noise intensity. Both conventional and stochastic resonances have been demonstrated experimentally. A wide class of sensory neurons demonstrates homoclinic chaotic spiking activity [17,25]. Coexistence of conventional and stochastic resonances may be significant for information processing in biological systems, since noise enhances both sensitivities to amplitude and frequency of the external signals.

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