Noise-Induced Phase Synchronization and Synchronization Transitions in Chaotic Oscillators

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Whether common noise can induce complete synchronization in chaotic systems has been a topic of great relevance and long-standing controversy. We first clarify the mechanism of this phenomenon and show that the existence of a significant contraction region, where nearby trajectories converge, plays a decisive role. Second, we demonstrate that, more generally, common noise can induce phase synchronization in nonidentical chaotic systems. Such a noise-induced synchronization and synchronization transitions are of special significance for understanding neuron encoding in neurobiology.

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The subject of this Letter is at the borderline of two basic families of phenomena in nonlinear systems nowadays attracting significant interest: noise-induced effects and synchronization. Noise-induced order was first reported [1] on the map which is directly connected to the Belousov-Zhabotinsky chemical reaction [2]. There a small amount of noise may change a chaotic trajectory of the system into a state similar to a periodic orbit smeared with noise [1], which makes the largest nonzero Lyapunov exponent (LLE) negative, and leads to a slower decay of correlations and an improvement of state predictability [3]. A negative LLE means that in an ensemble of systems with identical laws of motion and *common noise,* such as the motion of floating particles on a surface of an incompressible fluid [4], the states in the phase space shrink into a single point [4,5]; i.e., noise induces complete synchronization (CS) in chaotic systems.

Common noise is also of great relevance to biological systems. In ecology, similar environmental shocks may be responsible for synchronization of different populations over a large geographical region [6,7]. In neural systems, different neurons connected to another group of neurons will receive a common input signal which often approaches a Gaussian distribution as a result of integration of many independent synaptic currents [8]. It is especially important to emphasize experimental observations illustrating the remarkable reliability of repetitive spike sequence in neucortical neurons [8] in the response to repeated fluctuating stimuli that resemble real synaptic currents, a feature which is not observed in the response to constant input currents. From the viewpoint of common noise, repeatable firing means common synaptic current induces CS in neurons. This behavior is of great importance for the information processing of neurons: (1) single neurons may faithfully encode temporal information in the timing of successive spikes, (2) a group of neurons can respond collectively to a common synaptic current due to synchronization.

The effect of common noise on CS of identical chaotic systems was reconsidered [9], which spurred a long-standing dispute on the general conclusion that strong enough noise is able to synchronize chaotic systems. Some authors [10,11] found that synchronization of the logistic maps in Ref. [9] is an artifact due to finite precision in numerical simulations. Others claimed that it is the nonzero mean of the applied noise that plays a decisive role and an unbiased noise cannot lead to synchronization [12]. This claim, however, has been disproved by recent examples where unbiased noise indeed induces CS [13]. In fact, nonzero mean of noise can be viewed as an additional parameter which may move the system into another dynamical regime. So far, the mechanism of CS of chaotic systems induced by common (unbiased) noise is not clear and will be addressed in this Letter.

Moreover, real systems are typically nonidentical, and exact CS cannot be observed. It has not been explored whether noise can at least induce a weaker degree of synchronization in nonidentical systems. A typical weaker degree of synchronization in coupled nonidentical systems is generalized synchronization [14] and phase synchronization (PS) [15]. It has been recently found for coupled chaotic systems that before the transition to CS [16], there is the regime of PS associated with the transition of a zero Lyapunov exponent to negative values [15,17].

In this Letter, we study synchronization of two chaotic systems subjected to a common additive noise, i.e., $\dot{x} = f(x) + \xi$ and $\dot{y} = f(y) + \xi (f : R^n \rightarrow R^n)$ by examining both the LLE (λ_1) and the *zero Lyapunov exponents* (λ_2). A small initial difference $\delta x = y - x$ evolves approximately according to $\delta \dot{x} = Df(x)\delta x$. Note that the same Jacobian matrix $Df(x)$ governs the linearized dynamics in the noise-free case, but the trajectories in the phase space are different from the noisy case. This property is different from CS of coupled systems where $Df(x)$ is modified by coupling so that the synchronization manifold, which is the same as in the noncoupled system, becomes transversely stable [16]. Based on the linearized dynamics, Lyapunov exponents are well defined [18] similar to the deterministic case. Without noise, $\lambda_1 > 0$, because the autonomous system is chaotic, and λ_1 may become negative and synchronization occurs when the noise modifies considerably the dynamical structure in the phase space. Thus noise-induced CS should be understood from the inherent structure of $Df(x)$. Furthermore,

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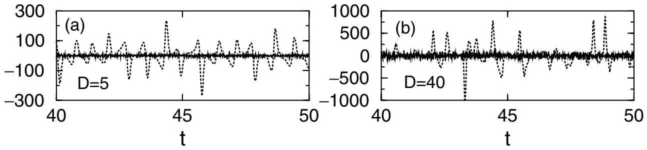

FIG. 1. Comparison between the velocity $f(x)$ and noise. Results are from the y component of the Lorenz system [Eq. (2)] with noise intensity D. For rather weak noise ($D = 5$) (a), the dynamical velocity $f_y = \rho x - y - xz$ (dotted line) is much larger in magnitude than noise most of the time. Even for rather large noise $D = 40$ (b), still there are long periods during which the noise is negligible.

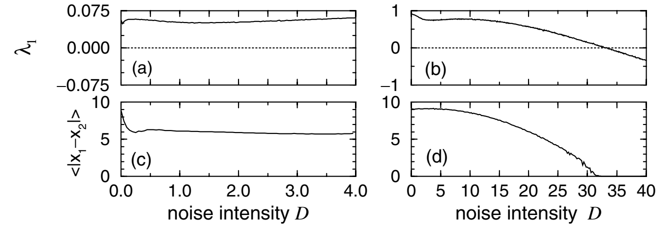

FIG. 2. The largest Lyapunov exponents λ_1 and average synchronization error $|x_1 - x_2|$ via noise intensity D. (a), (c) The Rössler system. (b), (d) The Lorenz system.

without noise, there is $\lambda_2 = 0$ associated with the perturbation δx along the trajectory; in phase coherent chaotic systems δx can be uniquely transformed to a relative phase $\delta\phi$ [15], which is marginally stable, so that a uniform distribution of initial phases in an ensemble of oscillators will remain uniform. For weak noise, most of the time it is $|f(x)| \gg |\xi|$ (see Fig. 1a), and we can also roughly speak of a motion along the trajectory and connect the original zero Lyapunov exponent to it and link it to the phase dynamics which often can be defined practically similar to the noise-free case. $\lambda_2 < 0$ means a preferred phase in an ensemble of oscillators; however, due to the stochastic character of the systems, a coherent phase corresponding to λ_2 cannot be rigorously defined, and we can expect only PS in a statistical sense. Defined in the above, the Lyapunov exponents of stochastic systems measure the sensitivity to perturbations of the initial condition, but not that to the driving noise [17]; as a result, they are generally no longer connected to the complexity of the systems [19].

We present our findings on the Rössler system

$$\dot{x} = -\omega y - z, \qquad \dot{y} = \omega x + 0.15y + D\xi, \qquad \dot{z} = 0.4 + z(x - 8.5), \tag{1}$$

with $\omega = 0.97$, and the Lorenz system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = \rho x - y - xz + D\xi, \qquad \dot{z} = -bz + xy, \tag{2}$$

with parameters $\sigma = 10$, $\rho = 28$, and $b = 8/3$. The noise ξ is a Gaussian one with $\langle \xi(t)\xi(t-\tau)\rangle = \delta(\tau)$.

I. Complete synchronization of identical systems with common noise.—As seen in Figs. 2(a) and 2(c), in the Rössler system, λ_1 keeps positive till the systems become unstable for $D > 4$, and the synchronization error is well above zero. However, in the Lorenz system, λ_1 becomes negative at rather strong noise ($D_c = 33.3$), and the synchronization error vanishes after D_c.

It is important to note that, in the Lorenz system, even for rather strong noise, the basic "butterfly" structure is preserved (Fig. 3). The systems explore a larger region of the phase space with increasing D. In the CS regime $D > D_c$, the trajectory is much more complex than a smeared periodic orbit and is quite different from the external noise [also see Fig. 1(b)]. Since now Lyapunov exponents are no longer a proper measure of complexity of the systems [19], common noise-induced synchronization is not necessarily linked to noise-induced order, as claimed in Ref. [12].

We understand different synchronization behavior of the Rössler and the Lorenz systems from the structure of their Jacobian matrix $Df(x)$. In the Lorenz system, there coexist a saddle point $S = (0, 0, 0)$ and two unstable foci C_1 and C_2 whose stable and unstable manifolds shape the geometry of the chaotic attractor. There exists a significantly large *contraction region* close to the stable manifold of S where $\mathrm{Re}(\Lambda_i) < 0$ $(i = 1, 2, 3)$ [Λ_i are the eigenvalues of $Df(x)$], and it is not frequently visited by unperturbed chaotic trajectories (Fig. 3). The saddle's two-dimensional stable manifold crossing the z axis separates the phase space into two halves [20]. Trajectories coming close to this stable manifold near the saddle point are rather possible to cross the stable manifold to enter into the same half-space and move in phase when receiving a common perturbation in the y direction. For large enough noise, the trajectories explore deep into the contraction region; λ_1 becomes negative and CS occurs when the contraction dominates over the expansion close to the unstable manifold of S. We note that the expansion region always exists, so that CS may be lost intermittently especially for D shortly beyond D_c, when there is additional perturbation from parameter mismatches or discrepancies between the two driving noises, which are inevitable in real systems. Perturbations to x or to both x and y have similar effects and CS can be expected and is verified numerically. However, due to the symmetry of the Lorenz system, noise acting only on the z direction does not have the tendency of bringing trajectories into the same half-space, and CS is not

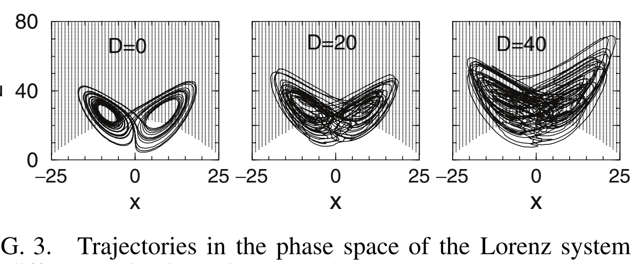

FIG. 3. Trajectories in the phase space of the Lorenz system at different noise intensity. The dotted background shows the contraction region in the plane $y = 0$.

observed accordingly. In the Rössler system, a contraction region with all three $\mathrm{Re}(\Lambda_i) < 0$ exists, but the contraction is very weak because the largest $\mathrm{Re}(\Lambda_i)$ is close to zero. In addition, in the presence of noise the systems still spend only a small portion of time in the contraction region. Contraction is not sufficient to induce CS. There are also regions in the phase space where all $\mathrm{Re}(\Lambda_i) > 0$, and strong enough noise ($D > 4$) makes the system access to such regions and breaks the system down easily.

This comparison study has clarified the mechanism of noise-induced CS. The existence of a significant contraction region plays the decisive role. If there does not exist a contraction region, CS cannot be achieved by any additive common driving signal. A biased noise may be easier to induce CS when its mean value moves the dynamics to a stable regime [12] in the contraction region.

II. Phase synchronization of nonidentical systems with common noise.—To study PS due to noise, we consider two systems with small parameter mismatches so that phases are not synchronized without noise. For the Rössler system, we use $\omega_1 = 0.97$ and $\omega_2 = 0.99$, and in the Lorenz system, we fix $\sigma_1 = 10$, $\rho_1 = 28$, $\sigma_2 = 10.2$, and $\rho_2 = 28.5$; this slight parameter difference does not change the Lyapunov exponent spectra of the systems much. λ_2 becomes negative at a relatively small D value in both the Rössler and the Lorenz systems (Figs. 4a and 4b). A phase linked to λ_2 now cannot be rigorously defined as in the noise-free case. Nevertheless, we can practically calculate a phase variable as in the deterministic system [15,17]; e.g., $\phi(t) = 2\pi[k + (t - \tau_k)/(\tau_{k+1} - \tau_k)]$, $\tau_k < t < \tau_{k+1}$, where τ_k and τ_{k+1} are two successive crossings of a Poincaré section after cycling a reference point (unstable fixed point of the noise-free system). The phase, however, is no longer coherent, and it is impossible to observe perfect synchronization of phases ϕ_1 and ϕ_2, i.e., $|\phi_1 - \phi_2| <$ const [15]. We expect to observe preferred phase differences at least when λ_2 becomes appreciably negative. An approach to study phase synchronization behavior in stochastic systems is to compute the distribution of cyclic phase difference, $P(\Delta\phi)$, on $[-\pi, \pi]$ [21]. A peak in $P(\Delta\phi)$ manifests a preferred phase difference between the systems, i.e., we interpret PS in a statistical sense [22].

Without noise the two nonidentical Rössler systems are not phase synchronized and the phase difference $\theta = \phi_1 - \phi_2$ decreases almost monotonously. Hence, the distribution $P(\theta)$ is very close to a uniform one (Fig. 5b). In contrast for strong enough noise, where we also have $\lambda_2 < 0$, one observes many plateaus in the phase difference, i.e., many phase-locking epochs (Fig. 5a, $D = 3.0$), and this is reflected by a pronounced peak around $\theta = 0$ in $P(\theta)$ (Fig. 5d); this illustrates a noise-induced phase synchronization. For D close to the transition of λ_2, phase-locking epochs are not very pronounced, and the peak is lower and is not located around θ (Fig. 5c). Similar properties are observed in the Lorenz system.

To understand PS induced by common noise, we examine an approximate phase dynamics of the Rössler system obtained similarly as in [15], i.e., $\dot{\theta} = \Delta\omega + 2D\xi \sin\frac{\phi_1+\phi_2}{2} \sin\frac{\theta}{2} + F(A)$, where $F(A)$ denotes fluctuations coming from the amplitudes. In a simplified version $\dot{\theta} = \Delta\omega + \sqrt{2}D\xi \sin\frac{\theta}{2} + D_1\eta$, $F(A)$ is described as independent Gaussian noise η. The common noise term $\sqrt{2}D\xi \sin\frac{\theta}{2}$ has a nonzero mean value $\langle\sqrt{2}D\xi \sin\frac{\theta}{2}\rangle_\xi = \frac{D^2}{4}\langle\sin\theta\rangle_\xi$ [23], which gives rise to a systematic contribution to the average dynamics (with respect to ξ) of the system. As a zero-order approximation, we arrive at the effective equation $\dot{\theta} = \Delta\omega + \frac{D^2}{4}\sin\theta + D_1\eta$. The analysis shows that when two oscillators are forced by a common noise, their phases establish a relationship which is equivalent to the case that they are coupled and subjected to perturbations; this zero-order approximation yields qualitatively the same features observed in the Rössler oscillators (Fig. 6).

We measure the degree of PS quantitatively by mutual information between the cyclic phases

$$M_1 = \sum_{i,j} p(i,j) \ln \frac{p(i,j)}{p_1(i)p_2(j)}, \tag{3}$$

where $p_1(i)$ and $p_2(j)$ are the probabilities when the phases ϕ_1 and ϕ_2 are in the ith and jth bins, respectively, and $p(i,j)$ is the joint probability that ϕ_1 is in the ith bin *and* ϕ_2 in the jth bin. The number of bins of $[-\pi, \pi]$ in our simulations is $N = 100$. M_1 is normalized into $[0,1]$ as $M = M_1/S_m$, where $S_m = \ln N$ is the Shannon

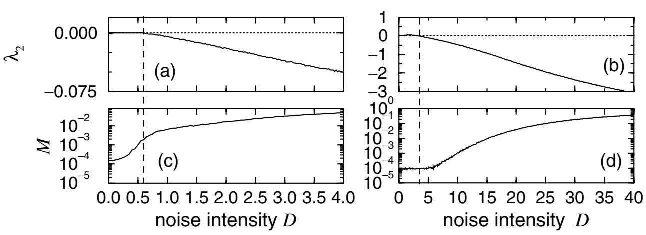

FIG. 4. The second Lyapunov exponent λ_2 and mutual information M via the noise intensity D. (a), (c) The Rössler systems. (b), (d) The Lorenz systems.

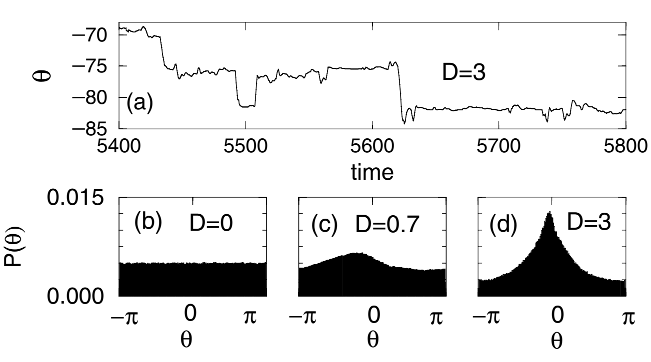

FIG. 5. (a) Time series of phase difference of two nonidentical Rössler systems ($\omega_1 = 0.97$ and $\omega_2 = 0.99$), at noise intensity $D = 0$ and $D = 3$. Distribution $P(\theta)$ of cyclic phase difference for $D = 0$ (b), $D = 0.7$ (c), and $D = 3$ (d).

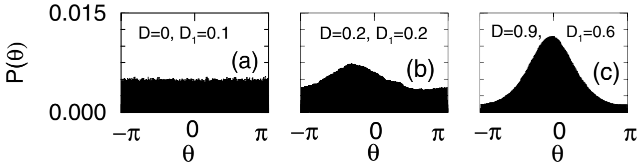

FIG. 6. Distribution $P(\theta)$ in the zero-order approximation of the phase model with $\Delta\omega = 0.02$, at different D and D_1 values.

entropy of the uniform distribution p_1 and p_2 (Figs. 4c and 4d). Because of the incoherence of the phases, an exact correspondence between the transition of λ_2 and PS would not be expected. Nevertheless, when λ_2 becomes appreciably negative, M increases rapidly, indicating an increasing degree of PS.

It is important to stress that noise-induced PS is observed in many perturbation configurations both in the Rössler and the Lorenz systems. This demonstrates that PS is more general. By PS, noise induces an observable macroscopic mean field in a large ensemble of elements. At weak noise intensity, this collective motion has small amplitude because the degree of synchronization is weak, while at large intensity, it is fairly noisy; it becomes the most coherent at a certain intermediate noise intensity, as is similar to other noise-induced resonantlike behavior [24]. This enhanced collective response to external fluctuating signals by synchronization may be of great importance in biology.

In summary, we have clarified the mechanism leading to CS by a common additive noise. A necessary condition for CS is the existence of a significant contraction region in the phase space, which in time-continuous systems may result from the existence of a saddle. Since there is clear evidence that a saddle point(s) underlies the firing mechanism in many neurons, such as in electroreceptors from dogfish and catfish, and from facial cold receptors [25], noise-induced synchronization is especially significant for interpreting experimental observations [8] and bringing a new understanding of neuron encoding. We have also demonstrated that noise is able to induce statistical phase synchronization in chaotic systems. Our results suggest a connection between phase synchronization and the transition of the zero Lyapunov exponent whose interpretation is generally not clear in stochastic systems [18]. In ecology, population dynamics may be well modeled by Rössler type chaotic dynamics [26], and noise-induced PS is significant for understanding the role of common environmental fluctuations on population synchronization [7].

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