Noise-Enhanced Phase Synchronization of Chaotic Oscillators

Changsong Zhou,[1] Jürgen Kurths,[1] István Z. Kiss,[2] and John L. Hudson[2]

[1]*Institute of Physics, University of Potsdam PF 601553, 14415 Potsdam, Germany*

[2]*Department of Chemical Engineering, 102 Engineers' Way, University of Virginia, Charlottesville, Virginia 22904-4741*

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The effects of noise on phase synchronization (PS) of coupled chaotic oscillators are explored. In contrast to coupled periodic oscillators, noise is found to enhance phase synchronization significantly below the threshold of PS. This constructive role of noise has been verified experimentally with chaotic electrochemical oscillators of the electrodissolution of Ni in sulfuric acid solution.

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The study of synchronization in coupled oscillators is of fundamental importance with applications in various fields [1]. The notation of synchronization has been extended to include a variety of phenomena in the context of interacting chaotic oscillators, such as complete synchronization (CS) [2], generalized synchronization [3], and phase synchronization (PS) [4]. PS can be achieved with a coupling strength much weaker than that for CS.

Noise influences synchronization in different ways. In CS of coupled chaotic systems, noise may induce intermittent loss of synchronization due to local instability of the synchronization manifold [5]. On the other hand, identical systems which are not coupled but subjected to a common noise may achieve CS at a large enough intensity, as has been demonstrated both in periodic [6] and chaotic systems [7]. The influence of random forcing on weakly coupled oscillators is of relevance in neuroscience [8] and ecology [9]. In PS of coupled oscillators, noise can induce phase slips in phase-locked periodic oscillators [10] and chaotic ones [11]. Internal noise-induced bursts in noncoupled sensories may achieve stochastic PS under a common external noise [12].

In this Letter we study the interplay between noise and weak coupling and report counterintuitive effects of noise on PS of coupled *chaotic* oscillators with different natural frequencies. Although noise induces phase slips in the phase-locked region, both independent and common noise can significantly *enhance* PS outside this region.

To take correlation of noise into account, we consider that the added noise $\sigma\xi_i(t)$ $(i = 1, 2)$ is composed of a common part $e(t)$ and an independent part $\eta_i(t)$, satisfying $\xi_i(t) = \sqrt{R}\, e(t) + \sqrt{1-R}\,\eta_i(t)$. Both $e(t)$ and $\eta_i(t)$ are assumed to be Gaussian noise and δ correlated in time.

In coupled periodic oscillators, $\Delta\dot{\phi} = \Delta\omega - \varepsilon\sin\Delta\phi$, perfect phase locking is achieved for $\varepsilon > \varepsilon_{ps} = \Delta\omega$. Noise smears out the border of the synchronization region which shrinks with increasing σ, as seen in Figs. 1(a) and 1(b) by the average frequency difference $\Delta\Omega = |\langle\Delta\dot{\phi}\rangle|$ in coupled Van der Pol oscillators. The degree of PS is slightly higher for larger noise correlation R, but the difference between independent noise $R = 0$ and common noise $R = 1$ is fairly small and becomes detectable only for rather strong noise. We thus see that noise degrades PS in coupled periodic oscillators, as has been previously shown for $R = 0$ [10].

We demonstrate quite different PS behavior in two coupled noisy chaotic Rössler oscillators

$$\dot{x}_{1,2} = -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \qquad (1)$$

$$\dot{y}_{1,2} = \omega_{1,2}x_{1,2} + 0.15y_{1,2} + \sigma\xi_{1,2}(t), \qquad (2)$$

$$\dot{z}_{1,2} = 0.4 + (x_{1,2} - 8.5)z_{1,2}, \qquad (3)$$

with $\omega_1 = 0.99$ and $\omega_2 = 0.97$. There are different ways to define a phase variable for a chaotic oscillator [13]. Recently, it has been rigorously shown that in phase coherent chaotic oscillations, there exists a transformation between the phases defined in different ways [13,14]. For the Rössler oscillator, it is convenient to introduce amplitude and phase variables as in Ref. [13], i.e., $A_i^2 = x_i^2 + y_i^2$ and $\tan\phi_i = y_i/x_i$. It has been shown that chaotic fluctuation of A_i introduces a noiselike perturbation to the dynamics of phase difference $\Delta\phi$, and PS in chaotic oscillators resembles that in noisy periodic ones [4,13]. At $\sigma = 0$, the transition point ε_{ps} is somewhat higher than that of the periodic oscillators (Fig. 1).

We show that adding some noise to the chaotic oscillators can *enhance* PS significantly. For $\sigma = 0.1$ [Fig. 1(c)], $\Delta\Omega$ is considerably smaller than that for $\sigma = 0$ indicating

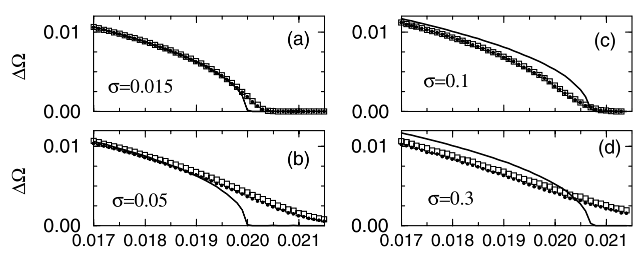

FIG. 1. Frequency difference $\Delta\Omega$ vs coupling strength ε. Solid lines: noise-free case $\sigma = 0$; open squares: independent noise $R = 0$; filled circles: common noise $R = 1$. Left panel: two coupled Van der Pol periodic oscillators $\ddot{x}_{1,2} - (1 - x_{1,2}^2)\dot{x}_{1,2} + \omega_{1,2}^2 x_{1,2} = \varepsilon(\dot{x}_{2,1} - \dot{x}_{1,2}) + \sigma\xi_{1,2}$ with $\omega_1 = 0.97$ and $\omega_2 = 0.99$; right panel: two coupled chaotic Rössler oscillators [Eqs. (1)–(3)] with $\omega_1 = 0.97$ and $\omega_2 = 0.99$.

enhanced PS below ε_{ps}; above ε_{ps} there is a small nonvanishing $\Delta\Omega$ as a result of noise-induced intermittent phase slips [11]. For $\sigma = 0.3$ [Fig. 1(d)], $\Delta\Omega$ is larger than that for $\sigma = 0$ around ε_{ps}; however, it is clearly smaller in weaker coupling strength, indicating enhanced PS. Similar to periodic oscillators, PS has only a rather weak dependence on the noise correlation R.

Figure 2(a) shows noise-enhanced PS for $\varepsilon = 0.0205$. At $\sigma = 0$, there are many epochs of phase synchronization between phase slips, and typically the epochs last for about 300 oscillation cycles. Adding a proper amount of noise to the two oscillators (e.g., $\sigma = 0.1$, $R = 1$) prolongates remarkably the duration of the synchronization epochs: the two oscillators maintain PS for a period of about 3000 oscillation cycles. However, for stronger noise (e.g., $\sigma = 0.3$, $R = 1$), phase slips occur more frequently again. To better characterize noise-enhanced PS, we focus on the mean duration $\langle\tau\rangle$ of the PS epochs. We find that $\langle\tau\rangle$ increases with the noise intensity σ, reaches a maximal value and decreases for larger σ for all coupling strengths analyzed [Fig. 2(b)]. The results are almost the same for independent noise $R = 0$, but at large σ, $\langle\tau\rangle$ takes slightly smaller values. Similar behavior has been observed close to the border of the synchronization region when ε is fixed, while $\Delta\omega$ is changed.

To understand this constructive effect of noise on PS, we examine how noise changes time scales of chaotic oscillations, since PS is essentially a phenomenon of adjusting time scales by weak interaction. We calculate the return time T between two successive returns of the chaotic trajectory to a Poincaré section. There are many repetitive configurations of T as a result of the fact that there are many unstable periodic orbits (UPOs) embedded in the chaotic attractors [15] and chaotic trajectories can stay close to a certain UPO for some time. When adding a small amount of noise to the system, e.g., $\sigma = 0.1$, the repetition has been reduced considerably, because noise prevents the system from following the UPOs closely for a long time. Noise may also speed up or delay the return of the orbits, thus generating both small and large return times not presented in the noise-free systems. The changes of time scales at weak noise are not clearly observable from the chaotic attractors.

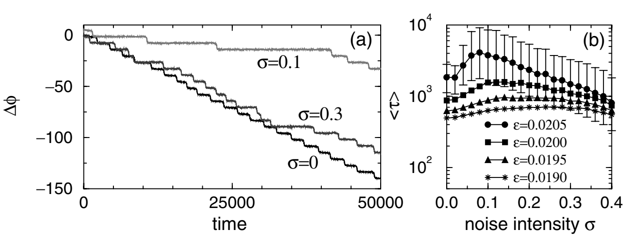

FIG. 2. Noise enhanced PS in two weakly coupled Rössler chaotic oscillators ($\varepsilon = 0.0205$). (a) Phase difference vs time for different noise intensity σ. (b) Average duration of PS epochs vs σ for different coupling strength ε. The standard deviation is shown with error bars for $\varepsilon = 0.0205$.

The interplay between noise and UPOs is important for understanding noise-enhanced PS. Similar to periodically driven chaotic oscillators [16], PS of coupled chaotic oscillators can be viewed as phase locking of a number of pairs of UPOs. Those pairs with larger difference in time scales generally achieve locking at a larger coupling strength. In the phase synchronization region, all pairs of UPOs are mutually locked. When the coupling strength is decreased past ε_{ps}, some pairs of UPOs become unlocked while others remain locked. In a pair of unlocked periodic orbits, the characteristic time for developing a phase slip has a dependence $\tau_{sl} \sim |\varepsilon - \varepsilon_{ps}|^{-1/2}$ as at typical type-I intermittency close to a saddle-node bifurcation [16]. Phase slips of a chaotic oscillation now become possible, but only when the system comes to follow one of the unlocked pairs for at least a time of τ_{sl} long enough for a phase slip to occur. In Fig. 3, we confirm that phase slips are indeed generated by unlocked UPOs, which is seen especially clearly for ε close to ε_{ps}, where only a few pairs of UPOs become unlocked and it takes a rather long time τ_{sl} to complete a phase slip. Periodic orbits are manifested by almost vanishing $\Delta X_k = |X_{n+k} - X_n|$, which is the difference between the x variable as every k returns to the Poincaré section $y = 0, x < 0$, with a return time T_k. It is seen that phase slips occur between a period-4 UPO in oscillator 1 and a period-2 UPO in oscillator 2 which are followed closely by the systems for a fairly long time (~30 cycles). While most orbits are locked with return times fluctuating around a common value ($T = 6.24$), these UPOs have clearly much smaller and larger return times [Figs. 3(e) and 3(f)], and thus remain unlocked by the coupling. With a noise of $\sigma = 0.1$, such a long time staying close to UPOs is rarely observed, and meanwhile most of the phase slips are eliminated (Fig. 2). At stronger noise, e.g., $\sigma = 0.4$, phase slips develop quickly when the oscillators come to some orbits with quite large differences in the return times, which cannot follow UPOs closely.

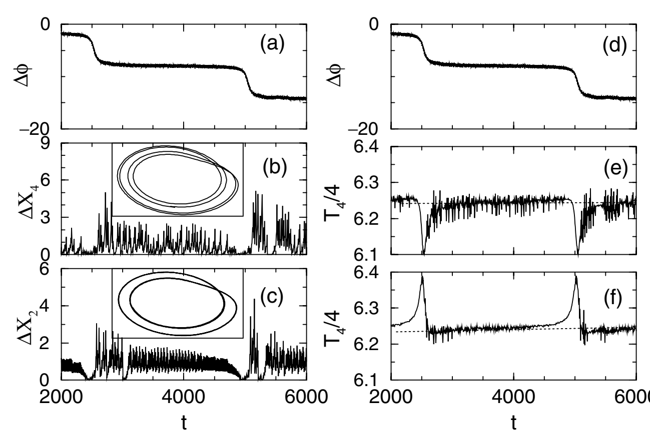

FIG. 3. Illustration of phase slips (a),(d) induced by unlocked UPOs; (b),(e) period-4 for oscillator 1; (c),(f) period-2 for oscillator 2 at $\varepsilon = 0.0205$. The insets in (b) and (c) show the unlocked UPOs around $t = 2500$. Many locked UPOs can also be observed in this presentation.

Thus we can explain noise-enhanced PS as follows. Noise has two effects: (i) it prevents the system from staying close to the unlocked UPOs for long enough times to allow a phase slip to occur and (ii) it generates fluctuation in the return times and may induce phase slips of locked orbits, as it does in coupled periodic orbits. The degree of PS is enhanced when (i) is dominant over (ii) at weak noise level, while it is degraded again when (ii) becomes dominant at large noise. There thus exists an optimal noise intensity yielding the maximal enhancement as a result of the competition between these two factors, as is similar to the resonantlike behavior typically observed between the interaction of noise and nonlinear systems [17]. At a smaller coupling strength ε, more orbits become unlocked, and phase slips may develop already during a shorter time τ_{sl} when the oscillators approach some unlocked orbits. When noise prevents a phase slip, the trajectories may approach other unlocked orbits quickly; thus the enhancement of PS becomes less pronounced [Fig. 2(b)]. For ε well below ε_{ps}, phase slips occur frequently and are not always clearly associated to UPOs. Beyond ε_{ps}, only (ii) is active, and perfect PS is interrupted by noise-induced phase slips. Increasing correlation R of noise can slightly enhance PS further.

We have shown that the interplay between noise and UPOs plays a constructive role in PS, which is in contrast to that close to the threshold of CS in coupled identical chaotic systems. There many UPOs are transversely unstable [5] and the synchronization error can be amplified from the noise level to generate bursts of desynchronization by these UPOs even in the presence of extremely weak independent noise components η [18].

We have also carried out experiments on noise-enhanced PS with two weakly coupled chaotic electrochemical oscillators. The reaction used is the electrodissolution of Ni in sulfuric acid solution. The oscillations result from the interaction of a (hidden) negative differential resistance of the faradaic process with potential drops in the electrolyte and/or in external resistances and with (normally slower) reaction or transport steps. By changing parameters such as applied potential, electrolyte concentration, and cell geometry steady, periodic, and chaotic behavior can be found. The reaction takes place on individual reacting sites (here two electrodes) and the currents, proportional to the rates of dissolution, can be independently measured. Such systems thus constitute a good platform for the study of coupled chaotic oscillators. In the experiments the two oscillators are chaotic and nonidentical. The oscillators are coupled through external resistors and common noise is added to the applied potential (driving force). Information dimension calculations and power spectrum and Hilbert phase analysis [19] showed that the chaos of each element is phase coherent and low dimensional. The electrodes are held at the applied potential $[V_{app}(t)]$ that is the sum of a constant potential (V_0 vs $Hg/Hg_2SO_4/K_2SO_4$ reference electrode) and a common noise, $\sigma e(t)$. Values of σ up to 2.0 mV are applied; inherent noise in the potential signal is less than 0.02 mV. In each experiment data are acquired, beginning approximately after 100 oscillations to assure stationarity, at 200 Hz. Further details of the experiments are given in Ref. [19].

The coupling strength between the two electrodes is varied with one series (R_s) and two parallel (R_p) resistors [19]. By keeping the total resistance $R_{tot} = R_s + R_p/2$ constant and changing the series resistance fraction, $\varepsilon = R_s/R_{tot}$, the coupling strength can be varied (no added coupling for $\varepsilon = 0$, and maximal added coupling for $\varepsilon = 1$). In a typical experiment PS sets in at about $\varepsilon_{ps} = 0.08$ without added noise. The transition to PS is sharp [20] which is a characteristic of systems with low levels of noise as can be seen in Fig. 1. CS can be observed at a much larger coupling strength of approximately $\varepsilon_{cs} = 0.8$ [19].

We first fix the coupling strength at $\varepsilon = 0.06$, i.e., just below ε_{ps}. The chaotic attractors of the two elements (not shown) are similar, but there is a small frequency mismatch ($\Delta\Omega = 0.005$ Hz). The two oscillators are not phase synchronized, as can be seen by the observed phase slip in Fig. 4(a). (The phases were calculated using the Hilbert-transform approach [13].) The analysis of the time series of the two oscillators shows that the phase slip occurs when both oscillators approach the neighborhood of an unlocked period-3 UPO [Fig. 4(b)]. The coincidence of the approach of unlocked UPOs and the phase slip confirms the numerical predictions about the dynamics close to but below ε_{ps}.

Noise-enhanced PS can be demonstrated at a somewhat lower coupling strength: $\varepsilon = 0.04$. During the time of the experiment (about 200 oscillations) there are two phase slips between the oscillators [see Fig. 5(a)] corresponding to a frequency difference of 0.012 Hz. As can be seen in Figs. 5(a)–5(c) the first phase slip can be attributed to the unlocked period-4 UPOs. The synchronization time τ_{sl} during the phase slip is much shorter than that for $\varepsilon = 0.06$. We see that the phase slips occur more frequently and develop more quickly than at the stronger coupling strengths. Moreover, the second phase slip cannot

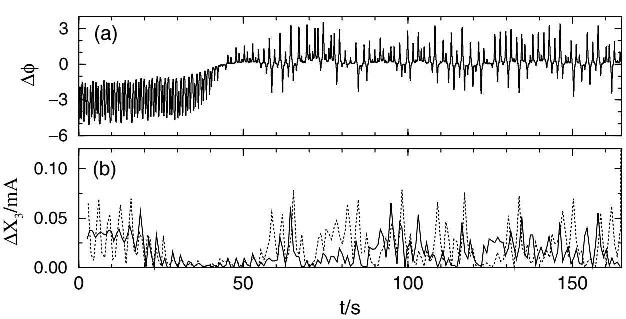

FIG. 4. Two ($\varepsilon = 0.06$) chaotic electrodes just below $\varepsilon_{ps} = 0.08$. $R_{tot} = 330\Omega$, $V_0 = 1.280$ V. (a) Phase difference between the two chaotic oscillators vs time. (b) The difference between the next return values of the current maxima ($\Delta X_3 = |X_n - X_{n-3}|$, where X_n is the nth maximum) of the two oscillators (solid line: oscillator 1; dashed line: oscillator 2) as a function of time.

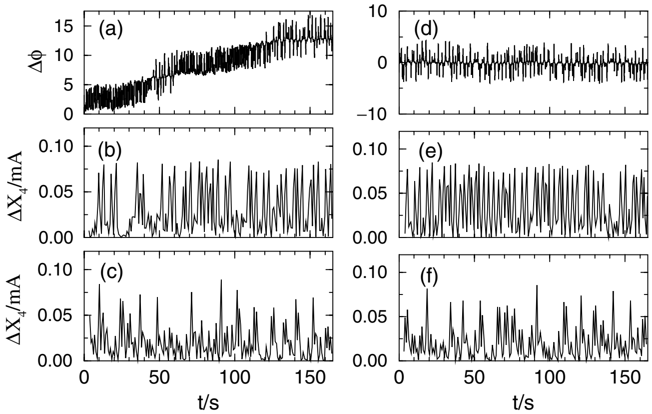

FIG. 5. Two coupled ($\varepsilon = 0.04$) chaotic oscillators without [left panel, (a)–(c)] and with [right panel, (d)–(f)] small amounts of common zero-mean Gaussian white noise (standard deviation of 0.3 mV measured at 200 Hz). Top row: phase difference between the two oscillators vs time. Middle and bottom rows: $\Delta X_4 = |X_n - X_{n-4}|$ (middle: oscillator 1, bottom: oscillator 2), vs time. $R_{\rm tot} = 500\ \Omega$, $V_0 = 1.350$ V.

be clearly linked to UPOs. These observations are also in agreement with the numerical calculations obtained further from $\varepsilon_{\rm ps}$.

By adding a small amount of zero-mean Gaussian white noise (the standard deviation is 0.3 mV measured at 200 Hz) to the (common) potential of the electrodes, we get a qualitatively different synchronization behavior. With this small noise, the deterministic nature of the electrodissolution process is still dominant; the reconstructed attractors (not shown) resemble those without noise. However, the phase slips are eliminated and the phase difference fluctuates around zero [Fig. 5(d)]. The oscillators do not have as long a time of residence close to UPOs [Figs. 5(e) and 5(f)] as in the noise-free case. The absence of phase slips during the 200 oscillations of the experiment is consistent with the model calculations which predict lengths of the PS epochs on the order of a thousand oscillations.

Experiments have also been carried out with weaker added coupling, $\varepsilon = 0.02$ and 0. No phase synchronization was obtained with noise up to an intensity at which the oscillators exhibited (noisy) periodic dynamics. With smaller electrode spacing and thus greater inherent coupling (4 mm rather than 18 mm as above), the added noise is able to achieve phase synchronization of the oscillators with less added coupling, $\varepsilon = 0.02$.

We have demonstrated both theoretically and experimentally that noise can play a very constructive role in the enhancement of phase synchronization of weakly coupled chaotic oscillators. Our finding is of significance for understanding the effect of fluctuations on synchronization in chemical, biological, and ecological systems. As an example of the latter, a combination of migration (weak coupling) and environmental fluctuations may be responsible for synchronization of populations [21].

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