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Doubly Stochastic Resonance

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We report the effect of doubly stochastic resonance which appears in nonlinear extended systems if the influence of noise is twofold: A multiplicative noise induces bimodality of the mean field of the coupled network and an independent additive noise governs the dynamic behavior in response to small periodic driving. For optimally selected values of the additive noise intensity stochastic resonance is observed, which is manifested by a maximal coherence between the dynamics of the mean field and the periodic input. Numerical simulations of the signal-to-noise ratio and theoretical results from an effective two state model are in good quantitative agreement.

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The subject of this Letter is at the borderline of two basic phenomena nowadays attracting significant interest of a broad readership. Both phenomena are marked out by the surprising ability of noise to create more order in the behavior of nonlinear systems when the intensity of the noise is increased. The first class of phenomena is noise-induced phase transitions, intensively investigated since the 1980s. Within the investigated models the appearance of new maxima in the system probability distribution, which has no counterpart in the deterministic description, has been observed [1]. The excitation of noise-induced oscillations [2,3] and the creation of a mean field in spatially extended systems [4–6] are further examples; various applications are discussed widely and a description of many other noise-induced behaviors, even of inhomogeneous structures, can be found in [1,4,6], and references therein.

The second basic phenomenon is stochastic resonance (SR) [7,8], which has been found in many natural systems [9]. The conventional situation is the Brownian motion in a bistable potential modulated by an external periodic force. For an optimally selected strength of noise, the Brownian particle hops coherently to the periodic input between the two wells. In addition to this situation, SR has been also found and investigated in a large variety of different classes of systems: monostable systems [10], systems with excitable dynamics [11], noisy nondynamical systems [12], systems without an external force [13], and systems without any kind of threshold [14].

However, SR has not been considered in systems with a noise-induced structure [15]. Therefore, we present in this Letter a new type of SR in a system with a noise-induced nonequilibrium phase transition resulting in a bistable behavior of the mean field. We call this effect *doubly stochastic resonance* (DSR) to emphasize that additive noise causes a resonancelike behavior in the structure, which in its own turn is induced by multiplicative noise.

This DSR is demonstrated on a nonlinear lattice of coupled overdamped oscillators first introduced in [5] and further studied in [6,16]. The following set of Langevin equations describes the considered system:

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i(t) + \frac{D}{2d}\sum_j (x_j - x_i) + \zeta_i(t) + A\cos(\omega t + \varphi), \quad (1)$$

where $x_i(t)$ represents the state of the ith oscillator, $i = 1, \ldots, L^d$, in the cubic lattice of the size L in d dimensions and with $N = L^d$ elements. The sum runs over $2d$ nearest neighbors of the ith cell, and the strength of the coupling is measured by D. The noisy terms $\xi_i(t)$ and $\zeta_i(t)$ represent mutually uncorrelated Gaussian noise, with zero mean and uncorrelated both in space and time

$$\langle \xi_i(t)\xi_j(t')\rangle = \sigma_\xi^2 \delta_{i,j}\delta(t - t'), \quad (2)$$

$$\langle \zeta_i(t)\zeta_j(t')\rangle = \sigma_\zeta^2 \delta_{i,j}\delta(t - t'). \quad (3)$$

The last item in (1) stands for an external periodic force with amplitude A, frequency ω, and initial phase φ.

For the sake of simplicity, the functions $f(x)$ and $g(x)$ are taken to be of the form [5]:

$$f(x) = -x(1 + x^2)^2, \qquad g(x) = 1 + x^2. \quad (4)$$

In the absence of external force ($A = 0$) this model can be solved analytically by means of a standard mean-field theory (MFT) procedure [4]. The mean-field approximation consists in replacing the nearest-neighbor interaction by a global term in the Fokker-Planck equation corresponding to (1). In this way, one obtains the following steady-state probability distribution w_{st}:

$$w_{\mathrm{st}}(x, m) = \frac{C(m)}{\sqrt{\sigma_\xi^2 g^2(x) + \sigma_\zeta^2}} \times \exp\left(2\int_0^x \frac{f(y) - D(y - m)}{\sigma_\xi^2 g^2(y) + \sigma_\zeta^2}\, dy\right), \quad (5)$$

where $C(m)$ is a normalization constant and m is a mean field, defined by the equation

$$m = \int_{-\infty}^{\infty} x w_{\mathrm{st}}(x, m)\, dx. \quad (6)$$

Solving Eq. (6) self-consistently with respect to the variable m one determines transitions between ordered ($m \neq 0$) and disordered ($m = 0$) phases. Transition boundaries between different phases are shown in Fig. 1 and the corresponding dependence of the order parameter on σ_ξ^2 is presented in Fig. 2. In addition to [5], we show influence of additive noise resulted in the shift of transition lines. For $\sigma_\zeta^2 = 0$ an increase of the multiplicative noise causes a disorder-order phase transition, which is followed by the reentrant transition to disorder [5]. In the ordered phase the system occupies one of two symmetric possible states with the mean fields $m_1 = -m_2 \neq 0$, depending on initial conditions.

Now let us turn to the problem of how system (1) responds to periodic forcing. We have taken a set of parameters $(\sigma_\xi^2; D)$ within the region of two coexisting ordered states with nonzero mean field. In particular, we choose values given by the dot in Fig. 1. As for the network, we take a two-dimensional lattice of $L^2 = 18 \times 18$ oscillators, which is simulated numerically [17] with a time step $\Delta t = 2.5 \times 10^{-4}$ under the action of the harmonic external force. The amplitude of the force A has to be set sufficiently small to avoid hops in the absence of additive noise during the simulation time of a single run which is much larger than the period of the harmonic force [18]. Jumps between $m_1 \leftrightarrow m_2$ occur only if additive noise is additionally switched on. Runs are averaged over different initial phases.

Time series of the mean field and the corresponding periodic input signal are plotted in Fig. 3 for three different values of σ_ζ^2. The current mean field is calculated as $m(t) = \frac{1}{L^2}\sum_{i=1}^N x_i(t)$. For a small intensity of the additive noise, hops between the two symmetric states m_1 and m_2 are rather seldom and not synchronized to the external force. If we increase the intensity σ_ζ^2, we achieve a situation when hops occur with the same periodicity as the external force and, hence, the mean field follows with high probability the input force. An increase of additive noise provides an optimization of the output of the system which is stochastic resonance. If σ_ζ^2 is increased further, the order is again destroyed, and hops occur much more frequently than the period of the external force.

Figure 3 illustrates that additive noise is able to optimize the signal processing in the system (1). In order to characterize this SR effect we have calculated signal-to-noise ratio (SNR) by extracting the relevant phase-averaged power spectral density $S(\omega)$ and taking the ratio between its signal part with respect to the noise background [8]. The dependence of SNR on the intensity of the additive noise is shown in the Fig. 4 for the mean field (filled points) and the mean field in a two-state approximation (opaque point). In this two-state approximation we have replaced $m(t)$ by its sign and put approximately $m(t) = +1$ or $m(t) = -1$, respectively. Both curves exhibit the well-known bell shaped dependence on σ_ζ^2 typically for SR. Since the bimodality of the mean field is a noise-induced effect we call that whole effect *doubly stochastic resonance*. For the given parameters and $A = 0.1$, $\omega = 0.1$ the maximum of the SNRs is approximately located near $\sigma_\zeta^2 \sim 1.8$.

Next we intend to give analytic estimates of the SNR. If A, D, and σ_ζ^2 vanish, the time evolution of the first moment of a single element is given simply by the drift part in the corresponding Fokker-Planck equation (Stratonovich case)

$$\langle \dot{x}\rangle = \langle f(x)\rangle + \frac{\sigma_\xi^2}{2}\langle g(x)g'(x)\rangle. \quad (7)$$

As it was argued in [6], the mechanism of the noise-induced transition in coupled systems can be explained by means of a short time evolution approximation [19]. It means that we start with an initial Dirac δ function, follow it only for a short time, such that fluctuations are small and the probability density is well approximated by a Gaussian. A suppression of fluctuations, performed by coupling, makes this approximation appropriate in our case [20]. The equation for the maximum of the probability, which is also the average value in this approximation $\bar{x} = \langle x\rangle$, takes the following form

$$\dot{\bar{x}} = f(\bar{x}) + \frac{\sigma_\xi^2}{2} g(\bar{x})g'(\bar{x}), \quad (8)$$

which is valid if $f(\langle x\rangle) \gg \langle \delta x^2\rangle f''(\langle x\rangle)$. For this dynamics an "effective" potential $U_{\mathrm{eff}}(x)$ can be derived, which has the form

$$U_{\mathrm{eff}}(x) = U_0(x) + U_{\mathrm{noise}} = -\int f(x)\,dx - \frac{\sigma_\xi^2 g^2(x)}{4}, \tag{9}$$

where $U_0(x)$ is a monostable potential and U_{noise} represents the influence of the multiplicative noise. In the ordered region, inside the transition lines (Fig. 1), the potential $U_{\mathrm{eff}}(x)$ is of the double-well form, e.g., $U(x)_{\mathrm{eff}} = -x^2 - 0.25x^4 + x^6/6$, for given $f(x)$, $g(x)$, and $\sigma_\xi^2 = 3$.

Now we consider a conventional SR problem in this potential with an external periodic force of the amplitude A and the frequency ω. If we neglect intrawell dynamics and follow linear response theory the SNR is well known [8,21]

$$\mathrm{SNR}_1 = \frac{4\pi A^2}{\sigma_\zeta^4}\, r_k, \tag{10}$$

where r_k is the corresponding Kramers rate [22]

$$r_k = \frac{\sqrt{(U''_{\mathrm{eff}}(x)|_{x=x_{\min}}|U''_{\mathrm{eff}}(x)|_{x=x_{\max}})}}{2\pi}\exp\left(-\frac{2\Delta U_{\mathrm{eff}}}{\sigma_\zeta^2}\right) \tag{11}$$

for surmounting the potential barrier ΔU_{eff}. Using Eqs. (9)–(11), we get an analytical estimate for a single element inside the lattice. Further on, rescaling this value by the number N of oscillators in the lattice [23] and taking into account the processing gain G and the bandwidth Δ in the power spectral density [21], the SNR_N of the mean field of the network of N elements can be obtained

$$\mathrm{SNR}_N = \mathrm{SNR}_1 \frac{NG}{\Delta} + 1. \tag{12}$$

This dependence is shown in Fig. 4 by the solid line and demonstrates, despite the rough approximation, a good agreement with the results of the numerical simulations. Nearly exact agreement is found in the location of the maximum as well as for the quantitative values of the SNR ("scalloping loss" [21] has been avoided in simulations by setting the frequency ω to be centered on one of the bins in the spectrum). A more satisfying theory of DSR is left as an open question in this Letter.

In conclusion, we have reported the existence of doubly stochastic resonance, which results from the twofold influence of noise on a nonlinear system. DSR is a combined effect which consists of a noise-induced phase transition and conventional SR.

Some remarks should be added. First, we have considered a system which undergoes a *pure* noise-induced transition, in the sense that a transition is impossible in the absence of noise. This is an important distinction of the DSR effect from SR in any variation of the mean-field model [24]. Second, in the considered system the so-called "stochastic" potential [1] for a single oscillator in the lattice [which differs from (9)] always remains monostable. Third, there are clear distinctions between SR and DSR behavior, because, in contrast to SR, in DSR additive noise does not only help an input/output synchronization, but also changes the properties of the system in the absence of the external force (see Figs. 1 and 2). As a consequence, in DSR amplitude of hops is decreased (bistability disappears) for large noise intensities σ_ζ^2 (compare Fig. 3 and Fig. 4 from [8]). Finally, not every system with noise-induced bistability demonstrates DSR, e.g., we did not find DSR in zero-dimensional systems, which are described in [1].

We expect that these theoretical findings will stimulate experimental works to verify DSR in real physical

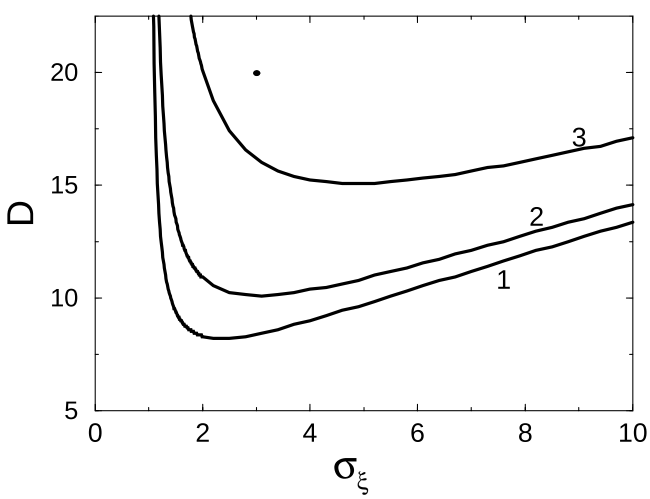

FIG. 1. Transition lines between ordered and disordered phase on the plane $(\sigma_m^2; D)$ for different intensities of the additive noise $\sigma_\zeta^2 = 0$ (1); 1 (2), and 5 (3). The black point corresponds to $D = 20$, $\sigma_\xi^2 = 3$.

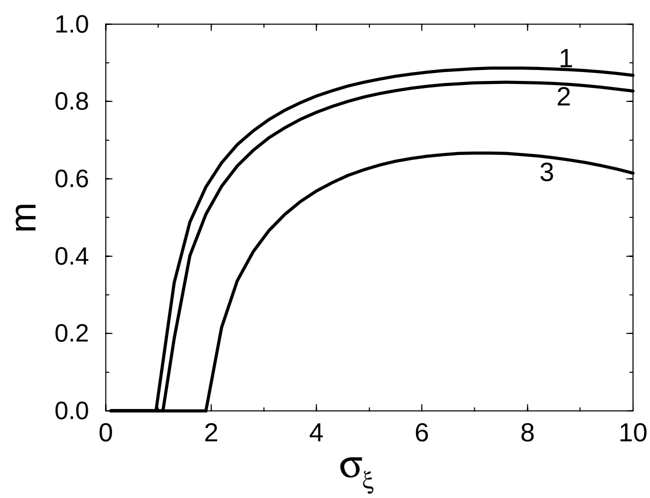

FIG. 2. The order parameter $|m|$ vs the intensity of multiplicative noise for $D = 20$ and $\sigma_\zeta^2 = 0$ (label 1), 1 (label 2), and 5 (label 3).

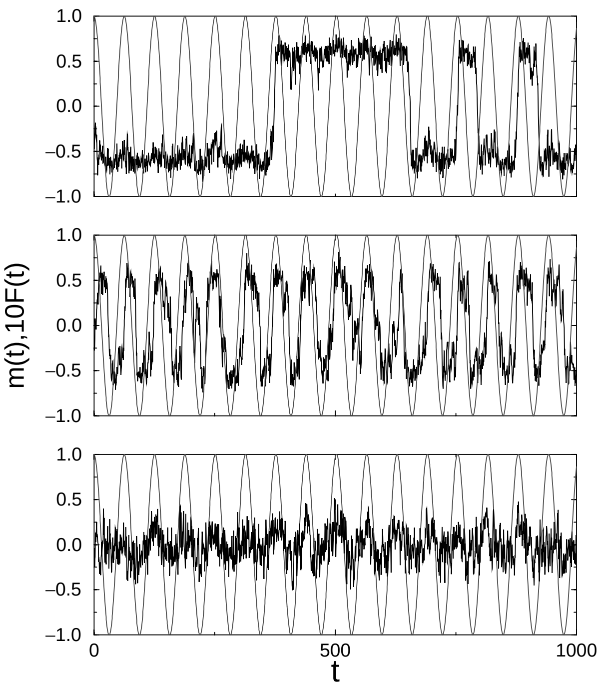

FIG. 3. Example of input/output synchronization. The time evolution of the current mean field (output) and the periodic external force $F(t)$ (input) for different intensities of additive noise (from top to bottom) $\sigma_\zeta^2 = 0.01$, 1.05, and 5.0. If the intensity of the additive noise is close to their optimal value (middle row), hops occur with the period of the external force. The remaining parameters are $A = 0.1$, $\omega = 0.1$, $D = 20$, and $\sigma_\xi^2 = 3$.

systems (for experiments on noise-induced bistability, see [25]). Appropriate situations can be found in electronic circuits [26], as well as in system, which demonstrate a

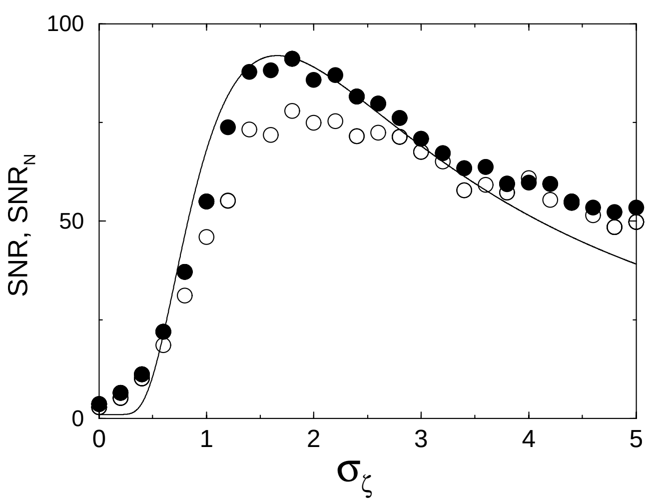

FIG. 4. The dependence of SNR on the additive noise intensity for the output (filled points) and its two-states approximation (opaque points). The solid line corresponds to the analytical estimation SNR_N (12), performed on the base of derivation of the "effective" potential and linear response theory. The parameters are the same as for Fig. 3 and the processing gain $G = 0.7$.

noise-induced shift of the phase transition, e.g., in liquid crystals [27], photosensitive chemical reactions [28], or Rayleigh-Bénard convection [29]. It can be crucial for such experiments that, in contrast to conventional SR, in DSR the energy of noise is used in a more efficient way: not only for the optimization of the signal processing, but also for the support of the potential barrier to provide this optimization.

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