Karthaus-2009 / Ice Sheets and Glaciers in the Climate System

Continuum mechanics

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Barnard Glacier, AK Austin Post Photo July 29, 1957

Kapitel 1

Vectors and tensors

1.1 Notation and the range, summation and comma conventions

1.1.1 Notation

Scalar variables are generally written in lower case letters, vectors in bold lower case, and matrixes in upper case bold letters. Thus, $\mathbf{a}$ denotes a vector, this vector has the length a (scalar variable). The components of the vector $\mathbf{a}$ with regard to some particular coordinate system are a_1, a_2, and a_3. The components of a matrix are generally written in lower case letters, for example the components of the matrix $\mathbf{A}$ are a_{ij}.

To denote a typical component of the $\mathbf{a}$ we write a_i where it is understood that i can stand for either 1, 2 or 3. We will also use the notation $[\mathbf{a}]_i$ to denote a typical component of $\mathbf{a}$, that is $a_i = [\mathbf{a}]_i$. Conversely, $[a_i]$ is taken to be the vector $\mathbf{a}$, that is $\mathbf{a} = [a_i]$. Correspondingly, if $\mathbf{A}$ is a matrix with the matrix elements a_{ij}, we have $[a_{ij}] = \mathbf{A}$ and $[\mathbf{A}]_{ij} = a_{ij}$.

We use $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ to denote the three unit vectors defining a Cartesian coordinate system. Note that the suffixes relate to the individual vectors and not to vector components. Sometimes the $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ are referred to as the x, y, and the z basis vectors of the coordinate system, and sometime we will refer to them as the first, second, and the third coordinate axis. This should cause no confusion.

The so called *Kronecker delta* δ_{ij} is a convenient quantity defined as

$$\delta_{ij} := \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Written in matrix notation the Kronecker delta is

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

showing that $[\delta_{ij}]$ is the unit matrix, and δ_{ij} the elements of the unit matrix. The symbol $\mathbf{I}$ stands for the unit matrix, and we have $[\mathbf{I}]_{ij} = \delta_{ij}$, and also $[\delta_{ij}] = \mathbf{I}$. It also follows, using our notation, that $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$, and that $[\hat{\mathbf{e}}_i]_j = \delta_{ij}$.

The *permutation symbol* ε_{ijk} is is defined as zero if two or more suffixes are equal. If not, it has either the value $+1$ or -1 depending on if the ijk is an even or odd permutation of 123.

Using this rule it, for example, follows that $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$, but $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$. To summarize, ε_{ijk} is defined as:

$$\varepsilon_{ijk} := \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of 123} \\ -1, & \text{if } ijk \text{ is an odd permutation of 123} \\ 0, & \text{if any two of } ijk \text{ are the same} \end{cases}$$

The permutation symbol is also known as the *Levi-Civita ε-Symbol*. It is, like δ_{ij}, often a very useful quantity. Using the Levi-Civita symbol, the *vector product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ can, for example, be written as

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a_i \, b_j \, \hat{\mathbf{e}}_k.$$

Another commonly used name for the vector product is the *cross product*. Furthermore we find that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \varepsilon_{ijk} a_i \, b_j \, c_k$$

from which it follows that

$$\varepsilon_{ijk} = \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) \, .$$

Later it will become somewhat easier to understand why exactly these two quantities are so useful. We will, for example, see that all isotropic tensors of second and third order can be written as $\alpha\delta_{ij}$ and $\beta\varepsilon_{ijk}$, respectively, where α and β are some constants.

A useful relationship between the Kronecker delta and the Levi-Civita symbol is the δ-ε relationship

$$\varepsilon_{ijk}\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq}. \tag{1.1}$$

1.1.2 Range convection

All suffixes take only the values 1, 2, and 3. This simple convention is known as the *range convention*. So for example the index i in x_i can only take the values 1, 2, and 3. Generally, x_i stands for the collection of the three quantities x_1, x_2, or x_3.

1.1.3 Summation convention

If a suffix is repeated once in a term, summation is taken over that suffix. For example if we write $c = a_i b_i$ it is to be understood that summation over the index i is implied, so that

$$c = \sum_{i=1}^{3} a_i \, b_i$$

Consider, for example, the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{33}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Using the range and summation convention this system of equations can be written as

$$a_{ij} \, x_j = b_i$$

Here, because the index j appears twice in the same term, summation over j is implied (summation convention). According to the range convention this sum is from 1 to 3. However, although the index i also appears twice, no summation over i is required because i does not appear twice in the **same** term.

Further examples are $a_i \, b_i$, which according to the summation convention is equal to $\sum_{i=1}^{3} a_i \, b_i$. On the other hand $a_i + b_i$ is not a sum according to the summation convention, because the index i is not repeated once in a single term. If a suffix is repeated more than once, the corresponding expressions is considered meaningless, and is not allowed under the summation convention. For example $a_i b_i c_i$ is not defined, and can not be used to represent the sum $\sum_{i=1}^{3} a_i \, b_i \, c_i$.

1.1.4 Comma convention

A partial derivative with respect to one of the spatial variables x_i is commonly written as

$$\frac{\partial}{\partial x_i}$$

So for example the partial derivative of a vector v_i with respect to the spatial variable x_j is

$$\frac{\partial v_i}{\partial x_j}$$

Under the range convention it is clear that the above expression stands for in total $3 \times 3 = 9$ quantities.

The comma convention states that a partial derivative of an arbitrary function with respect to one of the spatial variables x_j can be denoted through the index $,j$. Using this convention the above expression can, accordingly, be written as $v_{i,j}$, that is

$$\frac{\partial v_i}{\partial x_j} = v_{i,j}.$$

Using these conventions in combination often leads to considerable simplified and more compact notation. For example, the expression

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}$$

can be written using the range, summation and the comma notation as

$$v_{i,i}.$$

Exercise 1. *If* $\mathbf{f}(\mathbf{x})$ *is a vector function of the vector* $\mathbf{x}$*, show that*

$$\begin{array}{lrcl} \textit{(I)} & (x_k\, f_k)_{,i} & = & f_i + x_k f_{k,i} \\ \textit{(II)} & (x_k\, f_k)_{,ij} & = & f_{i,j} + f_{j,i} + x_k f_{k,ij} \end{array}$$

Solution.

$$\begin{array}{lrcl} \textit{(I)} & (x_k f_k)_{,i} & = & x_{k,i} f_k + x_k f_{k,i} \\ & & = & \delta_{ki} f_k + x_k f_{k,i} \\ & & = & f_i + x_k f_{k,i} \\ \textit{(II)} & (x_k f_k)_{,ij} & = & (f_i + x_k f_{k,i})_{,j} \\ & & = & f_{i,j} + x_{k,j} f_{k,i} + x_k f_{k,ij} \\ & & = & f_{i,j} + f_{j,i} + x_k f_{k,ij}\,. \end{array}$$

1.2 Vectors

A vector is a directed line element in space. A vector, thus, has a length and an orientation. If we have two vectors $\mathbf{u}$ and $\mathbf{v}$ the *dot* product of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and defined as

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta \tag{1.2}$$

where u and v are the lengths of the vectors $\mathbf{u}$ and $\mathbf{v}$, respectively, and θ is the angle between the vectors.

A vector is called a *unit vector* if its length is equal to unity. Two vectors are *orthogonal* if the angle between them is $\pi/2$. It follows that the dot product between two orthogonal vectors

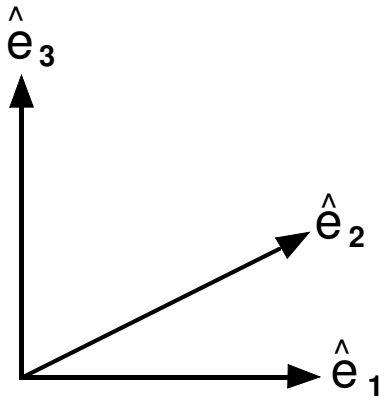

Abbildung 1.1: A Cartesian right-handed coordinate system

is equal to zero. A *Cartesian coordinate system* is defined through three unit *basis vectors* $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ which are mutually orthogonal, that is $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0$ for any $i \neq j$, and $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 1$ for $i = j$. We adopt a right-handed Cartesian coordinate system as standard. The coordinates of a vector $\mathbf{a}$ in the coordinate system defined by the three unit vectors $\hat{\mathbf{e}}_i$ for $i = 1, 2, 3$ are denoted by a_i where

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_2\hat{\mathbf{e}}_3.$$

If follows that the coordinates of the unit vector $\hat{\mathbf{e}}_1$, for example, are

$$\hat{\mathbf{e}}_1 = 1\hat{\mathbf{e}}_1 + 0\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3,$$

or simply $\hat{\mathbf{e}}_1 = (1, 0, 0)$

The components of any vector $\mathbf{a}$ in a coordinate system defined through the three orthonormal basis vectors $\hat{\mathbf{e}}_i$ for $i = 1, 2, 3$ can also be found by projecting $\mathbf{a}$ along the $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$. So for example

$$a_i = \mathbf{a} \cdot \hat{\mathbf{e}}_i. \tag{1.3}$$

Using the above given definition of the Kronecker delta δ_{ij} we can write

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}. \tag{1.4}$$

Exercise 2. *Show that in a coordinate system defined through the orthonormal basis vectors* $\hat{\mathbf{e}}_1$*,* $\hat{\mathbf{e}}_2$*, and* $\hat{\mathbf{e}}_1$ *we have*

$$[\hat{\mathbf{e}}_i]_p = \delta_{ip}.$$

Solution. *The expression* $[\hat{\mathbf{e}}_i]_p$ *stands for the* p *component of the unit vector* $\hat{\mathbf{e}}_i$*. Using Eqs. (1.3) and (1.4) we have*

$$[\hat{\mathbf{e}}_i]_p = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_p = \delta_{ip}.$$

Any vector $\mathbf{x}$ can be written in terms of the basis vectors as

$$\begin{aligned} \mathbf{x} &= x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \\ &= x_p\hat{\mathbf{e}}_p, \end{aligned} \tag{1.5}$$

where x_i for $i = 1, 2, 3$ are the *components* of the vector $\mathbf{x}$.

If we use a different set of basis vectors the components of a given vector with respect to that set of basis vector will, generally, be different. We can for example write

$$\begin{aligned} \mathbf{x} &= x'_1\hat{\mathbf{e}}'_1 + x'_2\hat{\mathbf{e}}'_2 + x_3\hat{\mathbf{e}}'_3 \\ &= x'_p\hat{\mathbf{e}}'_p \end{aligned} \tag{1.6}$$

where we also have but the set x'_i is different from x_i. Note that although the components of the vector $\mathbf{x}$ are different, that is $x_i \neq x'_i$, the vector, as an abstract quantity, has not changed so that

$$\mathbf{x} = x_i \hat{\mathbf{e}}_i = x'_i \hat{\mathbf{e}}'_i \tag{1.7}$$

If follows that

$$\mathbf{x} \cdot \hat{\mathbf{e}}_j = x_i \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = x_i \delta_{ij} = x_j.$$

If we have two vectors $\mathbf{x}$ and $\mathbf{y}$, the dot product of these two vectors is

$$\mathbf{x} \cdot \mathbf{y} = x_i \hat{\mathbf{e}}_i \cdot y_j \hat{\mathbf{e}}_j = x_i \, y_j \, \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = x_i \, y_j \, \delta_{ij} = x_i \, y_i.$$

proving the well known fact that

$$\mathbf{x} \cdot \mathbf{y} = xy \cos\theta = x_i \, y_i$$

It follows that

$$\mathbf{x} \cdot \mathbf{x} = x^2 = x_i \, x_i$$

or that $x = \sqrt{x_i \, x_i}$.

1.2.1 Coordinate transformations

We will now consider the relationships between the components of the same vector $\mathbf{x}$ in two different right-handed Cartesian coordinate systems. We refer to these two coordinates systems as system K and system K'. Two sets of orthonormal vectors: $\hat{\mathbf{e}}_i$, $\hat{\mathbf{e}}'_i$ define K and K', respectively Both coordinate system share the same origin.

As a first step we introduce the concept of *direction cosines* α_{ij} defined as

$$\alpha_{ij} := \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j = \underbrace{\cos(e'_i, e_j)}_{\text{Direction cosines}} \tag{1.8}$$

The nine quantities α_{ij} are, thus, defined as the dot products between the individual unit vectors of the two coordinate systems. Since these are unit vectors, it follows that α_{ij} is the cosine of the angel between the positive $\hat{\mathbf{e}}'_i$ axis and the positive $\hat{\mathbf{e}}_j$ axis.

Using the summation convention we have

$$\mathbf{x} = x_p \hat{\mathbf{e}}_p. \tag{1.9}$$

Taking the scalar product with $\hat{\mathbf{e}}'_i$ on both sides leads to

$$\begin{aligned} \mathbf{x} \cdot \hat{\mathbf{e}}_i &= x_p \hat{\mathbf{e}}_p \cdot \hat{\mathbf{e}}_i \\ &= x_p \delta_{pi} \\ &= x_i \end{aligned} \tag{1.10}$$

or

$$\begin{aligned} x_i &= \mathbf{x} \cdot \hat{\mathbf{e}}_i \\ &= x'_p \hat{\mathbf{e}}'_p \cdot \hat{\mathbf{e}}_i \qquad \text{(Siehe Gl. (1.6))} \\ &= x'_p \alpha_{pi}. \end{aligned} \tag{1.11}$$

Therefore

$$\boxed{x_i = \alpha_{pi} \, x'_p,} \tag{1.12}$$

giving us the relationship we were looking for between the coordinates of the vector in the two coordinate systems. It can furthermore be shown (exercise) that

$$\boxed{x'_i = \alpha_{ip} \, x_p.} \tag{1.13}$$

We now write

$$\begin{aligned} x'_i &= x_p\alpha_{ip} \qquad \text{(Equation (1.13))} \\ &= \alpha_{jp}x'_j\alpha_{ip} \qquad \text{(Equation (1.12))} \\ &= \alpha_{jp}\alpha_{ip}x'_j. \end{aligned}$$

and obtain $x'_i = \delta_{ij}x'_j$, so that

$$\delta_{ij}x'_j = \alpha_{jp}\alpha_{ip}x'_j,$$

or

$$\boxed{\alpha_{ip}\alpha_{jp} = \delta_{ij}}. \tag{1.14}$$

It can also be shown (exercise) that

$$\boxed{\alpha_{pi}\alpha_{pj} = \delta_{ij}}. \tag{1.15}$$

If we write

$$[\alpha_{ij}] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

expression (1.14) can be written as

$$\begin{aligned} \alpha_{ip}\alpha_{jp} &= \delta_{ij} \quad \text{oder} \quad [\alpha_{ij}][\alpha_{ij}]^T = 1, \\ \alpha_{pi}\alpha_{pj} &= \delta_{ij} \quad \text{oder} \quad [\alpha_{ij}]^T[\alpha_{ij}] = 1, \end{aligned}$$

and Eq. (1.12) as

$$\mathbf{x} = [\alpha_{ij}]^T\mathbf{x}' \tag{1.16}$$

showing that the matrix $[\alpha_{ij}]$ is a orthogonal matrix. The matrix $[\alpha_{ij}]$ is referred to as the *rotation matrix.*

Exercise 3. *Show using $x_i = \alpha_{ji}x'_j$ and $x'_i = \alpha_{ij}x_j$, that*

$$\alpha_{ij} = \frac{\partial x_j}{\partial x'_i} \tag{1.17}$$

and

$$\alpha_{ij} = \frac{\partial x'_i}{\partial x_j}. \tag{1.18}$$

Exercise 4. *The coordinate system K' is obtained by rotating the coordinate system K about the $\hat{\mathbf{e}}_3$ axis by the angle θ (See Fig. 1.2). Determine the components of the rotation matrix*

Solution. *The components of the rotation matrix are found by forming the appropriate dot products between the individual orthonormal unit vectors defining the two coordinate systems. These dot products can be written in a table of direction cosines as follows:*

$\cdot$	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\hat{\mathbf{e}}'_1$	$\cos\theta$	$\sin\theta$	0
$\hat{\mathbf{e}}'_2$	$-\sin\theta$	$\cos\theta$	0
$\hat{\mathbf{e}}'_3$	0	0	1

Using the well-known relationships

$$\begin{aligned} \cos(\theta - \pi/2) &= \sin\theta \\ \cos(\theta + \pi/2) &= -\sin\theta. \end{aligned}$$

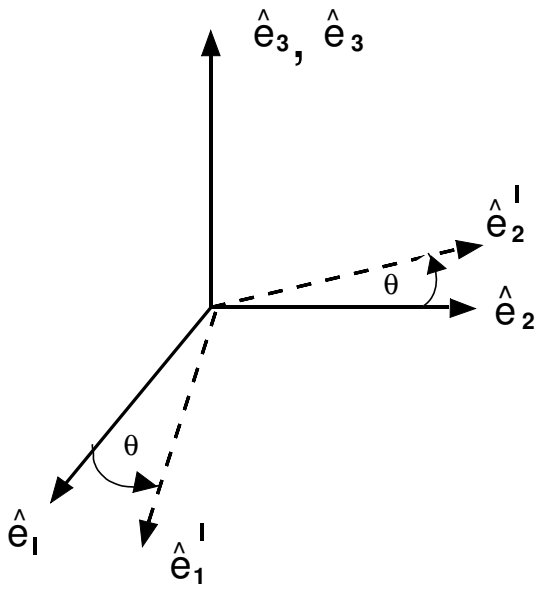

Abbildung 1.2: The coordinate systems K and K'

we have for example

$$\begin{aligned}\alpha_{21} = \cos(\hat{\mathbf{e}}_2', \hat{\mathbf{e}}_1) &= \cos(-\pi/2 - \theta) \\ &= -\sin\theta.\end{aligned}$$

and the rotation matrix is therefore

$$[\alpha_{ij}] = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 5. *In the coordinate system K the the vector $\mathbf{x}$ is given by $\mathbf{x} = 2\hat{\mathbf{e}}_1 + 1\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3$. Determine the components of $\mathbf{x}$ in the coordinate system K'.*

Solution. *We use the same rotation matrix as above and determine the components in K' using*

$$x_i' = \alpha_{ip}\, x_p.$$

It follows that

$$\begin{aligned} x_1' &= \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ &= \cos\theta \cdot 2 + \sin\theta \cdot 1 + 0 \cdot 3 \\ x_2' &= \alpha_{21}x_1 + \alpha_{22}\,x_2 + \alpha_{23}\,x_3 \\ &= -\sin\theta \cdot 2 + \cos\theta \cdot 1 + 0 \cdot 3 \\ x_3' &= \alpha_{31}\,x_1 + \alpha_{32}\,x_2 + \alpha_{33}x_3 \\ &= 0 \cdot 2 + 0 \cdot 1 + 1 \cdot 3, \end{aligned}$$

or

$$\mathbf{x} = (2\,\cos\theta + \sin\theta)\hat{\mathbf{e}}_1' + (-2\,\sin\theta + \cos\theta)\hat{\mathbf{e}}_2' + 3\,\hat{\mathbf{e}}_3'.$$

1.3 Cartesian tensors

We have shown that for a vector $\mathbf{a}$

$$\begin{aligned} \mathbf{a} &= a_p \hat{\mathbf{e}}_p \qquad \text{and} \\ \mathbf{a} &= a'_p \hat{\mathbf{e}}'_p \end{aligned} \tag{1.19}$$

and also that

$$\begin{aligned} a'_i &= \alpha_{ip} a_p \qquad \text{and} \\ a_i &= \alpha_{pi} a'_p. \end{aligned} \tag{1.20}$$

with α_{ij} being the components of the rotation matrix.

We now use these results and to **define** a quantity that we refer to as a *Cartesian tensor of order one.*

Definition. *A Cartesian tensor* $\mathbf{a}$ *of order one is an entity represented through three numbers referred to as the components of the tensor. For two different coordinate systems* K *and* K' *, the components obey the following transformation rules*

$$\begin{aligned} a'_i &= \alpha_{ip}\, a_p \qquad \textit{and} \\ a_i &= \alpha_{pi}\, a'_p, \end{aligned} \tag{1.21}$$

were α_{ij} *are the components of the corresponding rotation matrix.*

This definition can easily be extended. Consider the two Cartesian tensors $\mathbf{a}$ and $\mathbf{b}$ of order one. We have

$$\begin{aligned} b'_i &= \alpha_{ip} b_p \qquad \text{and} \\ b_i &= \alpha_{pi} b'_p \end{aligned}$$

and furthermore

$$\begin{aligned} a'_i b'_j &= (\alpha_{ip} a_p)(\alpha_{jq} b_q) \\ &= \alpha_{ip}\alpha_{jq} a_p b_q. \end{aligned}$$

This expression gives us the transformation rule for $a_i b_j$, which we can consider to be a new entity with nine components, as we go from system K to system k'

Let us now consider the inverse transformation rule.

$$\begin{aligned} a_i b_j &= (\alpha_{pi} a'_p)(\alpha_{qj} b'_q) \\ &= \alpha_{pi}\alpha_{qj} a'_p b'_q. \end{aligned} \tag{1.22}$$

If we define

$$c_{ij} := a_i b_j$$

in K and correspondingly define

$$c'_{ij} := a'_i b'_j$$

in K', we have shown that

$$c_{ij} = \alpha_{pi}\alpha_{qj} c'_{pq} \quad \text{and} \quad c'_{ij} = \alpha_{ip}\alpha_{jq} c_{pq}. \tag{1.23}$$

The entity $[a_i b_j]$ (or $[c_{ij}]$) can be written as a 3×3 Matrix,

$$[c_{ij}] := [a_i b_j] \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \tag{1.24}$$

Thus, from two vectors $\mathbf{a}$ and $\mathbf{b}$ we have defined a new entity $\mathbf{c} = [c_{ij}] := [a_i\, b_i]$ which can represented through a 3 matrix. We furthermore have shown that this entity follows the transformation laws (1.23).

Tabelle 1.1: Examples of vector and suffix notation

symbolic notation	suffix notation	
$c = \mathbf{a} \cdot \mathbf{b}$	$c = a_i b_i$	dot product
$c = \mathbf{A} : \mathbf{B}$	$c = a_{ij} b_{ij}$	double contraction
$\mathbf{c} = \mathbf{a} \times \mathbf{b}$	$c_i = \varepsilon_{ijk} a_j b_k$	vector product
$\mathbf{c} = \mathbf{A}\mathbf{b}$	$c_i = a_{ij} b_j$	matrix-vector multiplication
$\mathbf{C} = \mathbf{a} \otimes \mathbf{b}$	$c_{ij} = a_i b_j$	tensor product
$\mathbf{C} = \mathbf{A}\mathbf{B}$	$c_{ij} = a_{ik} b_{kj}$	matrix-matrix multiplication

Definition. *In a given coordinate system K the tensor product of two first order tensors* $\mathbf{a}$ *and* $\mathbf{b}$ *with the components a_i and b_i, is an entity with the components*

$$c_{ij} = a_i \, b_j.$$

In symbolic notation this relationship is expressed as

$$\mathbf{C} = \mathbf{a} \otimes \mathbf{b},$$

where the symbol $\otimes$ is used to denote the fact that $\mathbf{c}$ is the tensor product of $\mathbf{a}$ and $\mathbf{b}$. Some authors refer to $\mathbf{C}$ as the *dyad* of the vectors $\mathbf{a}$ and $\mathbf{b}$. Remember that in our notation we can refer to the components c_{ij} of a matrix $\mathbf{C}$ as $[\mathbf{C}]_{ij}$, and the components of a vector $\mathbf{a}$ as $[\mathbf{a}]_i$. We can therefore also write

$$[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i \, b_j = [\mathbf{a}]_i \, [\mathbf{b}]_j \tag{1.25}$$

Clearly, in general, $\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a}$ because generally $[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i \, b_j \neq b_i \, a_j = [\mathbf{b} \otimes \mathbf{a}]_{ij}$.

Note that for two vectors $\mathbf{a}$ and $\mathbf{b}$ we have now defined three different type of products: the scalar product ($\mathbf{a} \cdot \mathbf{b} = a_i b_i$), the vector product ($\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a_i \, b_j \, \hat{\mathbf{e}}_k$), and the tensor product ($[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i b_j$). As the names suggest, the scalar product is a scalar, the vector product a vector, and the tensor product a tensor.

We can define a further entity having the components

$$d_{ijk} := a_i \, b_j \, c_k$$

where a_i, b_i, and c_i are components of the vectors (first order tensors) $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The entity $\mathbf{d} = [d_{ijk}]$ is the tensor product of three vectors, that is

$$\mathbf{D} = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}.$$

Exercise 6. *What are the transformation rules for* $\mathbf{d}$ *defined as*

$$\mathbf{D} = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$$

where $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are first order tensors?*

Definition. *A Cartesian tensor* $\mathbf{a}$ *of order n is an entity represented through 3^n numbers referred to as the components of the tensor. For two different coordinate systems K and K' , the components obey the following transformation rules*

$$\begin{aligned} a'_{i_1,i_2,\ldots,i_n} &= \alpha_{i_1,j_1} \alpha_{i_2,j_2} \cdots \alpha_{i_n,j_n} a_{j_1,j_2,\ldots,j_n} \qquad \textit{and} \\ a_{i_1,i_2,\ldots,i_n} &= \alpha_{j_1,i_1} \alpha_{j_2,i_2} \cdots \alpha_{j_n,i_n} a_{j_1,j_2,\ldots,j_n} \end{aligned} \tag{1.26}$$

were α_{ij} are the components of the corresponding rotation matrix.

It follows from this definition that the tensor product of two tensors is a tensor.

Exercise 7. *Consider the scalar field $\phi = \phi(x_i)$. In any given coordinate system we define the components of the entity* $\mathbf{a}$ *as $a_i := \phi_{,i}$. Is* $\mathbf{a}$ *a tensor?*

Solution. *We must determine if the components of* $\mathbf{a}$ *follow the transformation rule (1.26). By definition we have in every coordinate system*

$$a'_i = \frac{\partial \phi}{\partial x'_i}.$$

Using the chain rule

$$a'_i = \frac{\partial \phi}{\partial x'_i} = \frac{\partial \phi}{\partial x_k}\frac{\partial x_k}{\partial x'_i} = \phi_{,k}\frac{\partial x_k}{\partial x'_i}$$

According to Eq (1.17)

$$\frac{\partial x_k}{\partial x'_i} = \alpha_{ik},$$

and therefore

$$a'_i = \alpha_{ik}\,\phi_{,k} = \alpha_{ik}a_k.$$

Comparing with (1.26) we see that a_i are the components of a first order tensor.

Exercise 8. *In one particular coordinate system K the components of a second order tensor* $\mathbf{A}$ *are*

$$\mathbf{A} = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We form the coordinate system K' through a positive rotation about the $\hat{\mathbf{e}}_3$ axis by the angle $\theta = \pi/4$. What are the components of $\mathbf{A}$ *in the K' system?*

Solution. *Since* $\mathbf{A}$ *is a second-order tensor it follows from (1.26) that*

$$a'_{ij} = \alpha_{ik}\alpha_{jl}a_{kl}. \tag{1.27}$$

The rotation matrix is

$$[\alpha_{ij}] = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\theta = \pi/4$. Inserting into (1.27) leads to

$$\begin{aligned} a'_{11} &= \alpha_{1k}\alpha_{1l}a_{kl} \\ &= \alpha_{11}(\alpha_{11}a_{11} + \alpha_{12}a_{12} + \alpha_{13}a_{13}) \\ &\quad +\alpha_{12}(\alpha_{11}a_{21} + \alpha_{12}a_{22} + \alpha_{13}a_{23}) \\ &\quad +\alpha_{13}(\alpha_{11}a_{31} + \alpha_{12}a_{32} + \alpha_{13}a_{33}) \\ &= \cos\theta(0\cos\theta + \sin\theta\gamma + 0) \\ &\quad +\sin\theta(\cos\theta\gamma + 0\,\sin\theta + 0) \\ &\quad +0(...) \\ &= 2\gamma\cos\theta\sin\theta = 2\gamma\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \gamma\,, \end{aligned}$$

and therefor

$$a'_{11} = \gamma\,.$$

We also find

$$\begin{aligned} a'_{12} &= \alpha_{1k}\alpha_{2l}a_{kl} \\ &= \alpha_{11}(\alpha_{21}a_{11}+\alpha_{22}a_{12}+\alpha_{23}a_{13}) \\ &\quad +\alpha_{12}(\alpha_{21}a_{21}+\alpha_{22}a_{22}+\alpha_{23}a_{23}) \\ &\quad +\alpha_{13}(\alpha_{21}a_{31}+\alpha_{22}a_{32}+\alpha_{23}a_{33}) \\ &= \cos\theta\,(-0\,\sin\theta+\gamma\,\cos\theta+0) \\ &\quad +\sin\theta(-\sin\theta\gamma+0\,\cos\theta+0) \\ &\quad +0(...) \\ &= \gamma(\cos^2\theta-\sin^2\theta) \\ &= \gamma(\cos^2\theta-\sin^2\theta)|_{\theta=\frac{\pi}{4}}=0\,, \end{aligned}$$

and

$$a'_{21}=0\,.$$

The components of $\mathbf{A}$ *in the* K' *system are, thus*

$$\mathbf{A}=\begin{pmatrix} \gamma & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 9. *Show that* $\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_i=\mathbf{I}$*, where* $\hat{\mathbf{e}}_i$ *for* $i=1,2,3$ *are unit vectors,* $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j=\delta_{ij}$*, and* $\mathbf{I}$ *is the unit matrix.*

Solution. *By the definition of a tensor product, any particular component of* $\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_i$ *is given by*

$$[\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_i]_{pq}=[\hat{\mathbf{e}}_\mathbf{i}]_p\,[\hat{\mathbf{e}}_\mathbf{i}]_q.$$

Because $[\hat{\mathbf{e}}_i]_p=\delta_{ip}$ *(see exercise 2) we have*

$$[\hat{\mathbf{e}}_\mathbf{i}]_p\,[\hat{\mathbf{e}}_\mathbf{i}]_q=\delta_{ip}\,\delta_{iq}=\delta_{pq}$$

that is

$$\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_i=\mathbf{I}.$$

Exercise 10. *Show that*

$$(\mathbf{a}\otimes\mathbf{b})\,\mathbf{c}=(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$$

Solution. *We first observe that* $\mathbf{a}\otimes\mathbf{b}$ *is a tensor product of two first-order tensors, and therefore a second-order tensor with the components* $[\mathbf{a}\otimes\mathbf{b}]_{ij}=a_i\,b_j$*. We can define* $\mathbf{D}$ *having the components* d_{ij} *as* $d_{ij}:=a_i\,b_j$ *and write* $(\mathbf{a}\otimes\mathbf{b})\mathbf{c}=\mathbf{D}\mathbf{c}$*, which represents the multiplication of the matrix* $\mathbf{D}$ *with the vector* $\mathbf{c}$*, the result being a vector. Now that we have seen that the expression* $(\mathbf{a}\otimes\mathbf{b})\mathbf{c}$ *is a vector we determine its components as follows*

$$[(\mathbf{a}\otimes\mathbf{b})\mathbf{c}]_i=d_{ij}\,c_j=a_i\,b_j\,c_j=a_i(\mathbf{b}\cdot\mathbf{c})=[(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}]_i$$

where we have used that $b_jc_j=\mathbf{b}\cdot\mathbf{c}$*. We thus have*

$$(\mathbf{a}\otimes\mathbf{b})\,\mathbf{c}=(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}.$$

Exercise 10 shows that the tensor product $\mathbf{a}\otimes\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ can be thought of as a linear transformation that transforms a vector $\mathbf{c}$ into another vector with the direction $\mathbf{a}$ according to the rule

$$(\mathbf{a}\otimes\mathbf{b})\,\mathbf{c}=(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}.$$

1.4 Invariants

A scalar quantity, associated with a tensor, that has the same value in all coordinate systems is called an *invariant* of the tensor. The length of a vector is, for example, the same in all coordinate systems and therefore an invariant of the vector.

Exercise 11. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are two vectors, show that their dot product is an invariant.*

Solution. *We must show that the product* $a_i b_i$ *is independent of the orientation of the coordinate system. Let* $c = a_i b_i$ *in some coordinate system* K*, and* a'_i*,* b'_j *be the components of* $\mathbf{a}$ *and* $\mathbf{b}$*, respectively, in coordinate system* K'*. The question is therefor, is* c' *defined as* $a'_i b'_i$ *equal to* c*? Using the fact that* $\mathbf{a}$ *and* $\mathbf{b}$ *are vectors and that their components therefore must follow the transformation rules (1.6) and (1.13) we find*

$$\begin{aligned} c' &= a'_i b'_i \\ &= \alpha_{ip} a_p \alpha_{iq} b_q \\ &= \alpha_{ip}\alpha_{iq} a_p b_q \\ &= \delta_{pq} a_p b_q \\ &= a_p b_p \\ &= a_i b_i \\ &= c. \end{aligned}$$

Exercise 12. *The second-order tensor* $\mathbf{C}$ *has, in one particular coordinate system, the components* c_{ij}*. Show that the trace of* $\mathbf{C}$ *given by* $\mathrm{tr}\mathbf{C} = c_{ii}$ *is an invariant of* $\mathbf{C}$*.*

Solution.

$$\begin{aligned} c'_{ii} &= \alpha_{ip}\alpha_{iq} c_{pq} \\ &= \delta_{pq} c_{pq} \\ &= c_{pp} = c_{ii} \end{aligned}$$

Exercise 13. $\mathbf{C}$ *is a second-order tensor. Show that* $c_{ik}\, c_{ki}$ *is an invariant.*

Solution.

$$\begin{aligned} c'_{ik} c'_{ki} &= \alpha_{iq}\alpha_{kp} c_{qp} \alpha_{kr}\alpha_{is} c_{rs} \\ &= \alpha_{iq}\alpha_{is}\alpha_{kp}\alpha_{kr} c_{qp} c_{rs} \\ &= \delta_{qs}\delta_{pr} c_{qp} c_{rs} \\ &= c_{qp} c_{pq} = c_{ik} c_{ki} \end{aligned}$$

Exercise 14. *Show that* $\mathrm{tr}\mathbf{A}^2$*, and* $\mathrm{tr}\mathbf{A}^3$ *are invariants.*

It is clear that the sum of two invariants is also an invariant. So, for example, since $\mathrm{tr}\mathbf{A}$ and $\mathrm{tr}\mathbf{A}^2$ are invariants of $\mathbf{A}$, the quantity $(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2$ is also an invariant of $\mathbf{A}$. It can, furthermore, be shown, that a second-order tensor has only three (independent) invariants. There is a certain degree of arbitrariness inevitable in defining a particular set of three invariants. We could, for example, use $\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{A}^2$, and $\mathrm{tr}\mathbf{A}^3$ as our set of three invariants, but it is custom to refer to the following set of invariants of a second-order tensor as the *principal invariants* of $\mathbf{A}$:

$$I_{\mathbf{A}} := \mathrm{tr}\mathbf{A} \tag{1.28}$$

$$II_{\mathbf{A}} := \frac{1}{2}((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2) \tag{1.29}$$

$$III_{\mathbf{A}} := \frac{1}{6}((\mathrm{tr}\mathbf{A})^3 + 2\,\mathrm{tr}\mathbf{A}^3 - 3\,\mathrm{tr}\mathbf{A}^2\,\mathrm{tr}\mathbf{A}) \tag{1.30}$$

Exercise 15. *Write expressions (1.28) to (1.30) in suffix notation*

Solution.

$$I_{\mathbf{A}} = a_{ij}, \tag{1.31}$$

$$II_{\mathbf{A}} = \frac{1}{2}(a_{ij}a_{ij} - a_{ii}a_{jj}), \tag{1.32}$$

$$III_{\mathbf{A}} = \frac{1}{6}(a_{ii}a_{jj}a_{kk} + 2a_{ij}a_{km}a_{mi} - 3a_{ik}a_{ki}a_{jj}). \tag{1.33}$$

The second principial invariant can also be written as

$$II_{\mathbf{A}} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}. \tag{1.34}$$

This follows directly from the definition of $II_{\mathbf{A}}$. The third principial invariant can be written on the form

$$III_{\mathbf{A}} = \det \mathbf{A}. \tag{1.35}$$

Some tedious algebra is needed to show this and we obmit here the proof.

Note that the characteristic equation of $\mathbf{A}$ can be written in terms of the three principial invariants as

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{23} & a_{33} - \lambda \end{vmatrix} \tag{1.36}$$

$$= \lambda^3 - I_{\mathbf{A}} + II_{\mathbf{A}}\lambda - III_{\mathbf{A}}. \tag{1.37}$$

Exercise 16. *Using the definitions of the three principial invariants ((1.28) to (1.30)), show that* $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^3 - I_{\mathbf{A}} + II_{\mathbf{A}}\lambda - III_{\mathbf{A}}$.

The fact that the characteristic equation of a second-order tensor $\det(\mathbf{A} - \lambda\,\mathbf{I})$ can be written on the form $\lambda^3 - I_{\mathbf{A}} + II_{\mathbf{A}}\lambda - III_{\mathbf{A}}$ goes some way towards explaining why this particular set of invariants is referred to as the principal invariants.

1.5 Isotropic tensors

A tensor whose components are unchanged by any rotation of the coordinate system is *isotropic*. For an isotropic tensor of rank-n with the components $a_{ijk\ldots}$ in the coordinate system K and the components $a'_{ijk\ldots}$ in a rotated coordinate system K' we therefore must have have

$$a'_{ijk\ldots} = a_{ijk\ldots}.$$

The unit tensor $\mathbf{I}$ is an example of an isotropic tensor, as we can easily verify in the following manner. The components of $\mathbf{I}$ are δ_{ij} and using the transformation rules of second-order tensors we find

$$\begin{aligned} \delta'_{ij} &= \alpha_{ip}\alpha_{jq}\delta_{pq} \qquad \text{(Eq. (1.26))} \\ &= \alpha_{ip}\alpha_{jp} \\ &= \delta_{ij} \qquad \text{(Eq. (1.14))} \end{aligned}$$

or

$$\delta'_{ij} = \delta_{ij}$$

showing that the components of $\mathbf{I}$ are unchanged by a rotation of the coordinate system. $\mathbf{I}$ is therefore an isotropic tensor.

It can furthermore be shown that all isotropic second-order tensors can be written on the form $\alpha\,\mathbf{I}$, where α is some scalar, and all isotropic third-order tensors $\mathbf{A}$ on the form $[\mathbf{A}]_{ijk} = \alpha\,\varepsilon_{ijk}$, where again α is some scalar.

Every fourth-order isotropic tensor can be written as

$$[\mathbf{A}]_{ijkl} = \alpha\,\delta_{ij}\delta_{km} + \beta\,\delta_{ik}\delta_{jm} + \gamma\,\delta_{im}\delta_{jk} \tag{1.38}$$

where α, β and γ are scalars. There are no first-order isotropic tensors.

Exercises

Exercise 17. *Which of the following expressions are meaningful?*

1. a_{ii}
2. $a_{ii}b_i$
3. $a_{rs}b_{sr}$
4. $a_{ij}b_j$
5. $a_{ijk}b_{ik}$
6. $a_{ii}b_{ii}$
7. $a_{ii}b_{jj}$
8. $a_{ijk}b_{ijkl}$

Exercise 18. *If $a_{ij} = -a_{ji}$ show that $a_{ij}x_ix_j = 0$ for all x_i.*

Exercise 19. *Show that*

1. $\delta_{ij}\delta_{ij} = 3$
2. $\delta_{ij}\delta_{jk}\delta_{ik} = 3$
3. $\delta_{ik}\delta_{jm}\delta_{ij} = \delta_{km}$

Exercise 20. *Write following expressions in matrix notation.*

1. $\alpha_{ip}\alpha_{jp} = \delta_{ij}$
2. $\alpha_{pi}\alpha_{pj} = \delta_{ij}$
3. $a_{ij} = \alpha\delta_{ij}b_{kk} + \beta b_{ij}$

Exercise 21. *Show that $\alpha_{ij} = \frac{\partial x_i'}{\partial x_j}$ where α_{ij} are the components of the rotation matrix. Hint: Calculate the partial derivative $x_i' = \alpha_{ip}x_p$ with respect to x_j.*

Exercise 22. *A tensor $\mathbf{A}$ has the components a_{ij}. Show that a_{ii} is an invariant of $\mathbf{A}$.*

Exercise 23. *Which of the following expressions are correct?*

1. $\varepsilon_{ijk} = \varepsilon_{jki}$
2. $\varepsilon_{ijk} = \varepsilon_{kij}$
3. $\varepsilon_{ijk} = -\varepsilon_{ikj}$
4. $\varepsilon_{ijk} = \varepsilon_{jik}$
5. $\varepsilon_{ijk} = -\varepsilon_{jik}$

Exercise 24. *Show that for any vector $\mathbf{a}$ the quantities $\varepsilon_{ijk}a_k$ form the components of a second-order tensor.*

Exercise 25. *If $\mathbf{A}$ with the components a_{ijkl} is a fourth-order tensor, show that*

$$b_{ij} := a_{ippj}$$

form the components of a second-order tensor.

Exercise 26. *The components a_{ij} and b_{ij} of the second-order tensors $\mathbf{A}$ and $\mathbf{B}$ are related through*

$$a_{ij} = c_{ijkl}\, b_{kl}$$

with c_{ijkl} being the components of a fourth-order isotropic tensor. If $\mathbf{B}$ is symmetric, i.e. $b_{ij} = b_{ji}$ show that

$$a_{ij} = \lambda\, \delta_{ij} b_{qq} + 2\,\mu\, b_{ij}$$

where λ and μ are some scalars. (Use Eq. (1.38) to show this.)

1.6 Vector and tensor calculus

Tabelle 1.2: Divergence, gradient, and curl (rotation). ϕ is a scalar, $\mathbf{u}$ a vector with the components u_i, and $\mathbf{A}$ a second-order tensor with the components a_{ij}.

symbolic notation	suffix notation	
$\nabla\phi$	$[\nabla\phi]_i = \phi_{,i}$	gradient
$\nabla \cdot \mathbf{u}$	$\nabla \cdot \mathbf{u} = u_{i,i}$	divergence
$\nabla \times \mathbf{u}$	$[\nabla \times \mathbf{u}]_i = \varepsilon_{ijk} u_{j,k}$	curl (rotation)
$\mathbf{u} \cdot \nabla$	$\mathbf{u} \cdot \nabla = u_j \frac{\partial}{\partial x_j}$	
$\nabla\mathbf{u}$	$[\nabla\mathbf{u}]_{ij} = u_{i,j}$	gradient
$\nabla^2\mathbf{u}$	$[\nabla^2\mathbf{u}]_i = \nabla^2 u_i$	Laplacian
$\nabla\mathbf{A}$	$[\nabla\mathbf{A}]_{ijk} = a_{ij,k}$	gradient
$\nabla \cdot \mathbf{A}$	$[\nabla \cdot \mathbf{A}]_i = a_{ij,j}$	divergence
$\nabla \times \mathbf{A}$	$[\nabla \times \mathbf{A}]_{ij} = \varepsilon_{ipq} a_{jp,q}$	curl (rotation)
$\nabla^2\mathbf{A}$	$[\nabla^2\mathbf{A}]_{ij} = \nabla^2(a_{ij}) = \nabla^2[\mathbf{A}]_{ij}$	Laplacian

$$\begin{aligned}
[\nabla\mathbf{v}]_{ij} &= \frac{\partial v_i}{\partial x_j} = v_{i,j} \qquad \text{(tensor)} \\
[(\nabla\mathbf{v})\mathbf{a}]_i &= \frac{\partial v_i}{\partial x_j} a_j \qquad \text{(vector)} \\
\mathbf{a} \cdot ((\nabla\mathbf{v})\mathbf{a}) &= a_i \frac{\partial v_i}{\partial x_j} a_j \qquad \text{(scalar)}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\mathbf{a} \cdot (\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathbf{T}})\mathbf{a} &= \frac{1}{2} a_i \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) a_j \\
&= a_i \frac{\partial v_i}{\partial x_j} a_j
\end{aligned}$$

Chapter 2

Kinematics

2.1 Introduction

Continuum mechanics treats physical objects as a continuous distribution of matter. The particular atomic and molecular structures are ignored. Only the collective properties of large number of atomic particles, such as density and temperature, are studied. This macroscopic approach is useful as long as we are interested in regions of matter that are large compared to atomic distances.

Continuum mechanics encompasses both the study of fluids and solids. It can be divided into the study of i) motion and deformation (kinematics), ii) forces in continuum (kinetics), iii) conservation laws (mass, momentum, energy, increase of entropy), and iv) the relation between forces and deformation (constitutive equations). This is also roughly how continuum mechanics will be presented here.

2.2 Material particles

Kinematics is the study of the motion of objects without reference to the forces involved. We will look at how motion of material particles and the deformation of a continuum can be described.

- A *continuum* is a hypothetical body in which the matter is continuously distributed.
- A *material particle* is a point-like part of the continuum having spatial dimensions which are large compared to atomic distances but small compared to the overall size of the object.

A material particle will be treated as a mathematical point, and the continuum is a connected set of such particles. The position of a material particle at $t = 0$ is given by $\mathbf{r}^\circ$. The function $\mathbf{r}(\mathbf{r}^\circ, t)$ describes the movement of this material particle over time

$$\mathbf{r} = \mathbf{r}(\mathbf{r}^\circ, t). \tag{2.1}$$

We will refer to the material particle having the position $\mathbf{r}^\circ$ at $t = 0$ as the material particle $\mathbf{r}^\circ$, that is $\mathbf{r}^\circ$ is both the position of the particle at $t = 0$ and its name. At some late time the material particle will, in general, be at some other location in space (given by the function $\mathbf{r}(\mathbf{r}^\circ, t)$) but its name ($\mathbf{r}^\circ$) stays the same.

Definition: The name (label) of every material particle is its position at at $t = 0$.

Note that since the motion of the material particle $\mathbf{r}^\circ$ is described by the function $\mathbf{r}(\mathbf{r}^\circ, t)$, we have

$$\mathbf{r}^\circ = \mathbf{r}(\mathbf{r}^\circ, t = 0)$$

.

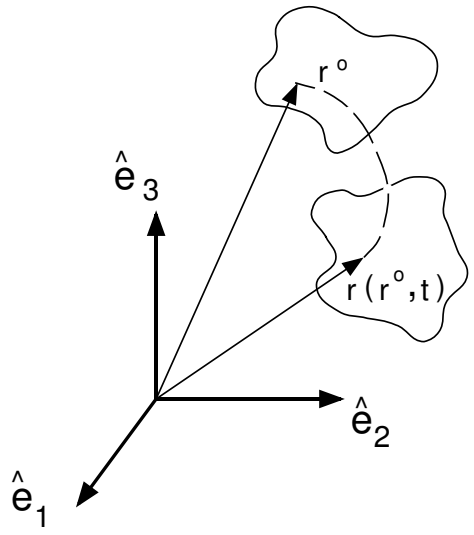

Figure 2.1: The motion of the material particle $\mathbf{r}^\circ$ as a function of time. Its position as a function of time is described by the function $\mathbf{r}(\mathbf{r}^\circ, t)$. The 'name' of this particular material particle is its location at $t = 0$, that is $\mathbf{r}^\circ = \mathbf{r}(\mathbf{r}^\circ, t = 0)$.

2.3 Material and local time derivatives

We must make a clear distinction between the rate-of-change of some variable (temperature, density, etc..c) at some specific point in space, and the rate-of-change of that variable as we follow a material particle on its journey through space. The temperature at some specific point in space may, for example, not change with time. If, however, the temperature changes from one point in space to another, the temperature of a material particle having a non-zero velocity will change with time.

Let the function φ describe some field variable (Temperature, Energy, Density). This function φ is then a function of $\mathbf{r}$ and t, with $\mathbf{r}$ being the position of the material particle $\mathbf{r}^\circ$,

$$\varphi = \varphi(\mathbf{r}(\mathbf{r}^\circ, t), t). \tag{2.2}$$

We define $\frac{D\varphi}{Dt}$ as the rate-of-change of φ for a given particle $\mathbf{r}^\circ$ (We can, for example, think of $D\varphi(\mathbf{r}(\mathbf{r}^\circ, t), t)/Dt$ as the rate-of-change of the temperature of the particle $\mathbf{r}^\circ$).

Definition: Let φ be some quantity of the material point $\mathbf{r}^\circ$. Then

$$\frac{D\varphi}{Dt}$$

is the rate-of-change of φ as seen by an observer following $\mathbf{r}^\circ$.

By definition, we therefore have

$$\frac{D\varphi}{Dt} := \left(\frac{\partial\varphi(\mathbf{r}(\mathbf{r}^\circ, t), t)}{\partial t}\right)\Bigg|_{\mathbf{r}^\circ=\text{fixed}}.$$

This derivative denotes the rate-of-change of the field variable φ as we follow the material particle $\mathbf{r}^\circ$ through space. This derivative is called the *material derivative* (German: *materielle* or the *substantielle Ableitung*).

The velocity $\mathbf{v}$ of the material particle $\mathbf{r}^\circ$ is the rate-of-change of the position of that particle. This means that we must consider the time derivative of $\mathbf{r}(\mathbf{r}^\circ, t)$ where $\mathbf{r}^\circ$ is kept fixed. The velocity is, thus, the material derivative of the position

$$\mathbf{v}(\mathbf{r}(\mathrm{r}^\circ, t), t) = \frac{D\mathbf{r}(\mathbf{r}^\circ, t)}{Dt} = \frac{\partial\mathbf{r}(\mathbf{r}^\circ, t)}{\partial t}\Bigg|_{\mathbf{r}^\circ=\text{fixed}}. \tag{2.3}$$

The *local time derivative* is the rate-of-change at a given point in space, that is

$$\frac{\partial \varphi}{\partial t} = \left(\frac{\partial \varphi(\mathbf{r}, t)}{\partial t} \right) \Big|_{\mathbf{r}=\text{fixed}}.$$

Exercise 27. *Think about the difference between the material and the local derivative. Can you think of an example where they are not equal?*

Since we have decided to express φ as a function of $\mathbf{r}$ and t, the local time derivative is simply the partial derivative with respect to t.

- The function $\varphi(\mathbf{r}, t)$ represents the value of φ at the point $\mathbf{r}$ and time t. A material particle situated at point $\mathbf{r}$ at the time t will experience this value of φ.
- The *material* time derivative $D\varphi/Dt$ is defined as the partial derivative of φ with respect to t with $\mathbf{r}^\circ$ held fixed. Physically, this is the rate-of-change of φ with time as a material particle is followed. For this reason the material derivative is sometimes also called the *derivative following a particle*.
- The *local* time derivative $\partial\varphi/\partial t$ is the partial derivative of φ with respect to t. It represents the rate-of-change of φ with time at the point $\mathbf{r}$.

Both the local and the material time derivatives are partial derivatives with respect to t. The essential difference is that the local derivative is defined for a function of $\mathbf{r}$ and t, whereas the material time derivative is defined for a function of the variables $\mathbf{r}^\circ$ and t. The form $\varphi(\mathbf{r}, t)$ is referred to as *spatial form* or as *Eulerian description* (German: *räumliche Darstellung* or *Eulersche Darstellung*). The form $\varphi(\mathbf{r}^\circ, t)$ is called *material description* or *Lagrangian description* (German: *materielle* or *Lagrangesche Darstellung*).

If the field variable φ is expressed in spatial form, that is

$$\varphi = \varphi(\mathbf{r}, t) = \varphi(r_i, t),$$

then we must express $\mathbf{r}$ as a function of $\mathbf{r}^\circ$ and t in order to calculate the material derivative. The function ϕ becomes a function or $\mathbf{r}$ and t, with $\mathbf{r}$ again being a function of $\mathbf{r}^\circ$ and t, that is

$$\varphi = \varphi(\mathbf{r}(\mathbf{r}^\circ, t), t) = \varphi(r_i(r_k^\circ, t), t).$$

Using the chain rule of partial differentiation we obtain

$$\underbrace{\left.\frac{\partial \varphi}{\partial t}\right|_{r_k^\circ}}_{\frac{D\varphi}{Dt}} = \underbrace{\left.\frac{\partial \varphi}{\partial r_i}\right|_t}_{\varphi_{,i}} \underbrace{\left.\frac{\partial r_i}{\partial t}\right|_{r_k^\circ}}_{v_i} + \underbrace{\left.\frac{\partial \varphi}{\partial t}\right|_{r_i}}_{\frac{\partial \varphi}{\partial t}},$$

or

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + v_i \varphi_{,i} \tag{2.4}$$

$$= \frac{\partial \varphi}{\partial t} + (\mathbf{v} \cdot \nabla)\varphi. \tag{2.5}$$

which can be written in the form

$$\boxed{\frac{D}{Dt} = \underbrace{\underbrace{\frac{\partial}{\partial t}}_{\text{local derivative}} + \underbrace{(\mathbf{v} \cdot \nabla)}_{\text{convective derivative}}}_{\text{material derivative}}}$$

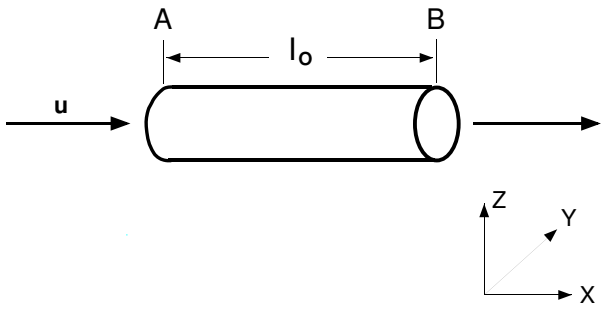

Figure 2.2: A fluid flows through the pipe with a velocity u. As it enters the pipe at the left end is has a somewhat higher temperature ($T = T_A$) than at exit from the right end ($T = T_B$). At every location within the pipe the temperature of the fluid does not change with time. The local derivative is, thus, zero. The temperature of a material particle, nevertheless, changes in the course of time.

The second term on the right-hand side is called the *convective* derivative (German: die *konvektive* Ableitung). The material derivative can be considered to be the sum of the local and the convective derivatives.

Remark: We have

$$\frac{D\varphi}{Dt} = 0 \iff \varphi \quad \text{(German: materiell konstant)}$$
$$\frac{\partial\varphi}{\partial t} = 0 \iff \varphi \quad \text{steady state.}$$

Exercise 28. *Can you think of an example where the material derivative is zero but the local time derivative not?*

Example: Consider a pipe filled with fluid (Fig. 2.2). The fluid moves with the velocity u through the pipe. The fluid enters the pipe at the left end having the temperature T_A. As it flows through the pipe it looses heat to the surroundings and the temperature decreases linearly with distance travelled through the pipe (Fig. 2.3). At exit through the right end it has the temperature $T_B \neq T_A$.

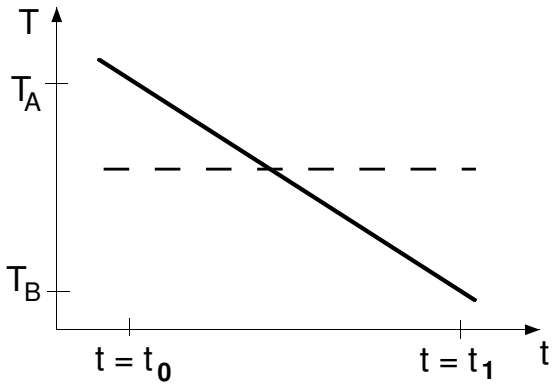

Figure 2.3: Temperature as a function of time at some particular point within the pipe (dashed line), and for a material particle (thick line).

For an observer who measures the temperature of the fluid at some point $\mathbf{r}$ within the pipe, the readings will not change with time (dashed line in Fig. 2.3). For an observer who follows a

material fluid particle and measures its temperature along the way, the readings will change with time (thick line in Fig. 2.3).

Exercise 29. *For $T = a + bx_1$ with $x_1 \in [0, 1]$ and $\mathbf{v} = (c, 0, 0)$ calculate the material derivative of T.*

Solution: By using Eq. (2.4) we obtain

$$\begin{aligned} \frac{DT}{\partial t} &= \frac{\partial T}{\partial t} + v_i T_{,i} \\ &= 0 + v_1 \frac{\partial T}{\partial x_1} + \frac{\partial T}{\partial x_2} v_2 + \frac{\partial T}{\partial x_3} v_3 \\ &= cb \end{aligned}$$

The **acceleration** of a material particle $\mathbf{r}^\circ$ is the material derivative of the velocity.

Exercise 30. *What is the physical meaning of the local derivative of the velocity? How is it different from the material derivative?*

We put $\varphi = v_i$ into Eq. (2.4) and obtain

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_k v_{i,k},$$

or

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}.$$

A velocity field is stationary if $\partial \mathbf{v}/\partial t = 0$, and uniform if $(\mathbf{v} \cdot \nabla)\mathbf{v} = 0$

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= 0 \quad \Longleftrightarrow \quad \mathbf{v} \text{ stationary} \\ (\mathbf{v} \cdot \nabla)\mathbf{v} &= 0 \quad \Longleftrightarrow \quad \mathbf{v} \text{ uniform.} \end{aligned}$$

2.4 Material surfaces

A *material surface* is a surface that for every t consists of the same material particles.

Question: Is the glacier surface a material surface?

Question: Can there be a flux of mass through a material surface?

Question: Can there be heat flux through a material surface?

2.4.1 Material surfaces and the material derivative

Let a material surface be defined by

$$F(\mathbf{r}, t) = 0 \tag{2.6}$$

We will now show that it then follows that $DF/Dt = 0$.

For $t = 0$ consider some point on a material surface defined by Eq. (2.6). Let the material particle at that point be called $\mathbf{P}^\circ$. For $t = 0$ we, thus, have $F(\mathbf{P}^\circ, 0) = 0$. Since the surface is a material surface, it must consist of the same material particles at all times. The particle $\mathbf{P}^\circ$ will, thus, stay on the surface forever and $F(\mathbf{P}(\mathbf{P}^\circ, t), t) = 0$ for all t. The rate-of-change of F as $\mathbf{P}^\circ$ is followed is therefore zero. This is true of any material particle on the surface so

$$\frac{DF}{Dt} = 0. \tag{2.7}$$

Conversely, let us now assume that $DF/Dt = 0$ and show that it then follows that $F(\mathbf{r}, t) = 0$ defines a material surface.

If $DF(\mathbf{r}, t)/Dt = 0$ it follows from the definition of D/Dt that the value of $F(\mathbf{P}(\mathbf{P}^\circ, t), t)$ does not change as some material particle $\mathbf{P}^\circ$ is followed, that is $DF(\mathbf{P}(\mathbf{P}^\circ, t), t)/Dt = K$ for every material particle $\mathbf{P}^\circ$, where K is some constant. The set of all such material particles fulfilling $F(\mathbf{r}, t) = K$ forms a surface for some t (This is a general mathematical fact, because for every t we have one equation for three unknowns.). The equation $F(\mathbf{r}, t) = K$, hence, defines a material surface for any value of K. Setting $K = 0$ concludes the proof.

We, thus, have seen that a surface described by an equation of the form $F(\mathbf{r}, t)$ is a material surface if and only if $DF/Dt = 0$.

$$\boxed{\frac{DF}{Dt} = 0 \quad \Longleftrightarrow \quad F(\mathbf{r}, t) = 0 \text{ defines a material surface}} \tag{2.8}$$

Equation (2.7) can also be written on the form

$$\frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla) F = 0$$

and also as

$$\frac{\partial F}{\partial t} + v_i F_{,i} = 0.$$

The unit normal $\hat{\mathbf{n}}$ to the surface is

$$\hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|}, \tag{2.9}$$

and it follows that

$$\begin{aligned} \frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla) F &= \frac{\partial F}{\partial t} + \mathbf{v} \cdot (\nabla F) \\ &= \frac{\partial F}{\partial t} + \mathbf{v} \cdot \hat{\mathbf{n}} |\nabla F| \\ &= 0. \end{aligned}$$

Thus

$$\mathbf{v} \cdot \hat{\mathbf{n}} = -\frac{\frac{\partial F}{\partial t}}{|\nabla F|}, \tag{2.10}$$

with $\mathbf{v} \cdot \hat{\mathbf{n}}$ being the speed of the surface in the direction of the normal $\hat{\mathbf{n}}$ to the surface.

Exercise 31. *Show that* $(\mathbf{v} \cdot \nabla) F = \mathbf{v} \cdot (\nabla F)$

Solution:

$$\begin{aligned} (\mathbf{v} \cdot \nabla) F &= \left((v_x, v_y, v_z) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \right) F \\ &= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) F \\ &= \left(v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} + v_z \frac{\partial F}{\partial z} \right) \\ &= (v_x, v_y, v_z) \cdot \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \\ &= \mathbf{v} \cdot (\nabla F). \end{aligned}$$

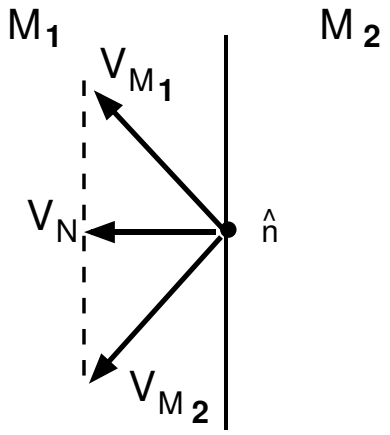

Figure 2.4: The solid line represents a material surface separating the two regions M_1 and M_2. At the point $\mathbf{P}$ the velocity field may be unsteady, and the tangential components $\mathbf{v}_{M_1}$ and $\mathbf{v}_{M_2}$ will in general not be equal. The velocity components normal to the surface $\mathbf{v}_{M_1}$ and $\mathbf{v}_{M_2}$ on the other hand must be equal.

2.4.2 Surface speed

The velocity component tangential to a material surface can be discontinuous across the surface. Let the two regions M_1 and M_2 be divided by a material surface with P being a material point on this surface.

$$\begin{aligned} \mathbf{v}_{M_1} &:= \lim_{\mathbf{x}\to\mathbf{P}} \mathbf{v}(\mathbf{x}), \quad \mathbf{x} \in M_1 \\ \mathbf{v}_{M_2} &:= \lim_{\mathbf{x}\to\mathbf{P}} \mathbf{v}(\mathbf{x}), \quad \mathbf{x} \in M_2 \\ \mathbf{v}^{\hat{\mathbf{n}}}_{M_i}(P,t) &:= \lim_{\mathbf{x}\to\mathbf{P}} \mathbf{v}(\mathbf{x},t)\cdot\hat{\mathbf{n}}, \quad \mathbf{x} \in M_i \quad \text{for} \quad i = 1,2 \end{aligned} \tag{2.11}$$

In general we have $\mathbf{v}_{M_1} \neq \mathbf{v}_{M_2}$. On the other hand $\mathbf{v}^{\hat{\mathbf{n}}}_{M_1} = \mathbf{v}^{\hat{\mathbf{n}}}_{M_2}$. The velocity

$$\mathbf{u} := \mathbf{v}^{\hat{\mathbf{n}}}_{M_1} = \mathbf{v}^{\hat{\mathbf{n}}}_{M_2}$$

is the velocity of the material surface. It is the speed with which the surface moves in a direction normal to itself. A material point at the surface can travel with a velocity different from the velocity $\mathbf{u}$ of the surface if it has a nonzero velocity component in tangential direction to the surface.

Exercise 32. *Give an example for a material surface where the tangential velocity is unsteady across the surface.*

2.5 Kinematics of the glacier surface

Let the vertical position of the glacier surface be described as a function of x and y

$$F(x,y,z,t) := z_s(x,y,t) - z = 0 \tag{2.12}$$

If there is no accumulation or ablation, the glacier surface is a material surface. In general this is, however, not the case. The accumulation/ablation rate along the surface will be described by the function $\dot{b}(x,y,t)$ in the units $\mathrm{m\,a^{-1}}$. Accumulation is defined to be positive and ablation negative.

Because the glacier surface is not a material surface the velocity of a material particle at the surface is, in general, not equal to the velocity of the surface itself (Fig. 2.5). We will denote the

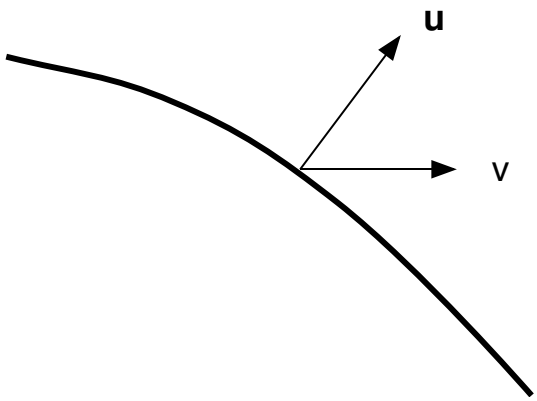

Figure 2.5: The vector $\mathbf{v}$ is the velocity of a material particle at the surface, and $\mathbf{u}$ is the velocity of the surface.

velocity of a material particle situated at some moment at the surface by $\mathbf{v}$, while $\mathbf{u}$ denotes the velocity of the surface. Only if

$$(\mathbf{u} - \mathbf{v}) \cdot \hat{\mathbf{n}} = 0$$

will $F(\mathbf{r}, t) = 0$ describe a material surface. If F does not describe a material surface we do, in general, not expect $(\mathbf{u} - \mathbf{v}) \cdot \hat{\mathbf{n}} = 0$, and for that reason

$$\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F \neq 0.$$

We have on the other hand

$$\frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F \ = 0. \tag{2.13}$$

This is shown in detail in an exercise below. Note that because u is not a material velocity, (2.13) does not correspond to the material derivative of F.

We define

$$\dot{b}_\perp := (\mathbf{v} - \mathbf{u}) \cdot \hat{\mathbf{n}} \tag{2.14}$$

were $\hat{\mathbf{n}}$

$$\hat{\mathbf{n}} = \frac{\nabla F}{|\nabla F|}$$

is the unit normal to the surface. Note that we can write the components of $\hat{\mathbf{n}}$ as

$$n_i = \frac{F_{,i}}{|\nabla F|}.$$

The function $\dot{b}_\perp$ is defined as the projection of the difference between the material velocity at the surface ($\mathbf{v}$) and the surface velocity ($\mathbf{v}$) onto the unit surface normal ($\hat{\mathbf{n}}$). (Question: What is $\dot{b}_\perp$ in figure 2.5?) If this difference is equal to zero the corresponding material point travels with the same speed in the direction normal to the surface as the surface itself. The material point remains at the surface and there is no flux of material across it. If on the other hand $\dot{b}_\perp \neq 0$ there must be a flux of material across the surface. The function $\dot{b}_\perp$ thus represents the normal flux through the surface.

From Eq. (2.13) it follows that

$$\frac{\partial F}{\partial t} + (u_i + v_i - v_i) F_{,i} = 0$$

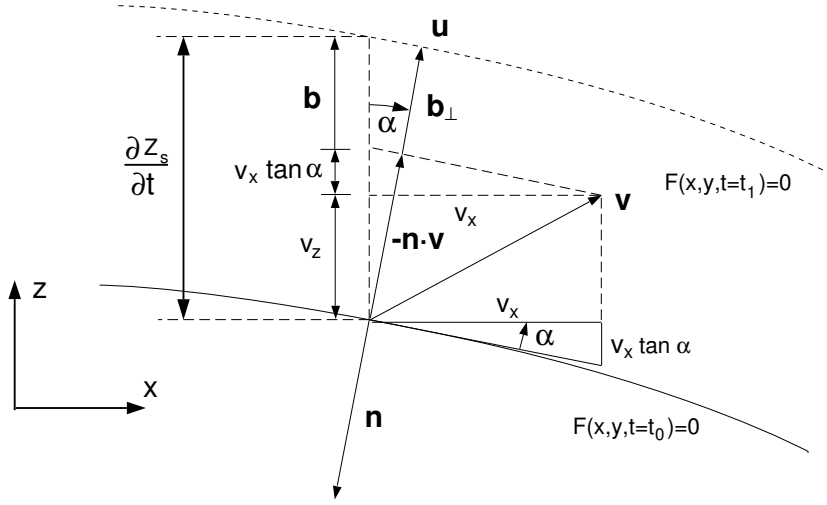

Figure 2.6: The individual components of the kinematic boundary condition.

or

$$\begin{aligned}\frac{\partial F}{\partial t} + v_i F_{,i} &= (v_i - u_i)F_{,i} \\ &= (v_i - u_i)n_i|\nabla F| \\ &= |\nabla F|\dot{b}_\perp.\end{aligned}$$

If we define

$$\dot{b} := |\nabla F|\dot{b}_\perp. \tag{2.15}$$

we can write this equation as

$$\boxed{\frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F = \dot{b}} \tag{2.16}$$

For

$$F(x,y,z,t) := z_s(x,y,t) - z = 0.$$

we obtain from Eq. (2.16)

$$\boxed{\frac{\partial z_s}{\partial t} + v_x\frac{\partial z_s}{\partial x} + v_y\frac{\partial z_s}{\partial y} - v_z = \dot{b}} \tag{2.17}$$

This equation describes the relation between the velocity components of a material particle at the surface (v_i), the surface slope ($\partial z_s/\partial x$ and $\partial z_s/\partial y$), the changes in altitude with time, and the mass-balance rate ($\dot{b}$, German: Die Massenbilanzfunktion) Eq. (2.16) is the so called *kinematic boundary condition at the surface*. The individual terms of Eq. (2.16) are shown graphically in Fig. (2.6).

From the definition (2.14) of $\dot{b}_\perp$ we see that $\dot{b}_\perp$ is the flux through the glacier surface as measured in the direction normal to the surface. The geometrical meaning of $\dot{b}$ as defined through (2.15) is possibly less clear. Figure 2.7 shows the graphical relationship between $|\nabla F|$, $|\partial F/\partial z|$, and $\dot{b}_\perp$. For a surface defined by $F(x,y,z,t) = 0$ the gradient of F is always normal to the surface. $\partial F/\partial z$ is the component of gradient of F in z direction. As described above, it follows from its definition, that $\dot{b}_\perp$ represents the material flux normal to the surface. $\dot{b}$ is defined as $\dot{b} := |\nabla F|\dot{b}_\perp$. If F is written on the form $F(x,y,z,t) := z_s(x,y,t) - z = 0$ we can give $\dot{b}$ a simple geometrical interpretation. Because now $|\partial F/\partial z| = 1$ we can write $\dot{b}_\perp/\dot{b} = |\partial F/\partial z|/|\nabla F|$ which shows that $\dot{b}_\perp$ is to $\dot{b}$ as $|\partial F/\partial z|$ is to $|\nabla F|$. In figure 2.7 it is assumed that F is written on this form, in which case $\dot{b}$ can be thought of as the flux through the surface as measured with respect to a horizontal plane.

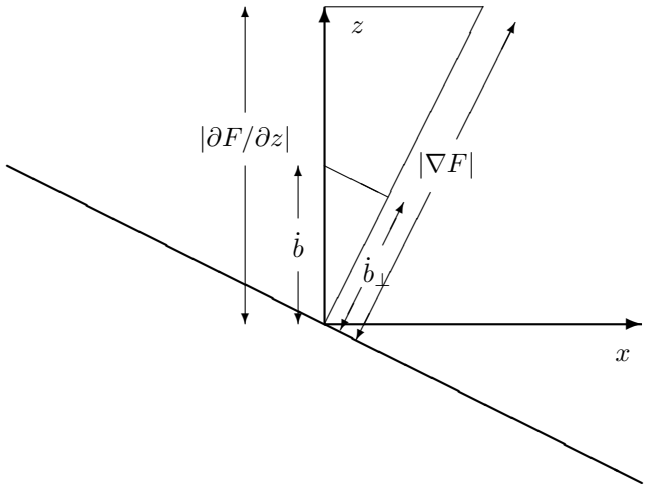

Figure 2.7: The geometrical relationship between $\dot{b}$ and $\dot{b}_\perp$.

Exercise 33. *If $F(\mathbf{r},t)=0$ is not a material surface but moves with a speed $\mathbf{u}$ different from the velocity $\mathbf{v}$ of the material particles at the surface, show that*

$$(\mathbf{v}-\mathbf{u})\cdot\hat{\mathbf{n}}=\frac{DF/Dt}{|\nabla F|}$$

where $\hat{\mathbf{n}}$ is the normal to the surface.

Exercise 34. *Why is Eq. 2.13 correct?*

Answer: For $t=0$ let $\mathbf{o}$ be some point at the surface. We define the velocity of this point to be the velocity of the surface at this point in space at all times. The point $\mathbf{o}$ will therefore always stay at the surface and its velocity will be equal to $\mathbf{u}$. The position of $\mathbf{o}$ can be described by the function $\mathbf{p}(\mathbf{o},t)$. We have

$$u_i=\left.\frac{\partial p_i}{\partial t}\right|_{o_k}$$

The surface is represented by the equation $F(\mathbf{r},t)=0$. Because $\mathbf{o}$ is always situated at the surface we have $F(\mathbf{p}(\mathbf{o},t),t)=0$ and

$$\left.\frac{\partial F}{\partial t}\right|_{\mathbf{o}}=0$$

Using the chain rule we obtain

$$\left.\frac{\partial F}{\partial t}\right|_{\mathbf{o}}=\left.\frac{\partial F}{\partial p_i}\right|_t\left.\frac{\partial p_i}{\partial t}\right|_{o_k}+\left.\frac{\partial F}{\partial t}\right|_{p_i}$$

and therefore

$$\frac{\partial F(r_i,t)}{\partial t}+u_k\frac{\partial F(r_i,t)}{\partial r_k}=0$$

which is Eq. (2.13).

Example: A surface is represented by the equation $F(\mathbf{r},t)=0$ with

$$F=x-ut.$$

The equations describes a surface moving with the velocity u in x direction. It consists of all points in the yz plane with $x = ut$ and we have

$$\frac{\partial F(r_i,t)}{\partial t} + u_k \frac{\partial F(r_i,t)}{\partial r_k} = -u + u = 0$$

Exercise 35. *What is the physical meaning of $\dot{b}$ as defined by Eq. (2.15)?*

Question: Does the correctness kinematic boundary condition depend on the material properties of the ice?

2.6 The kinematic boundary condition at bed

An analogous equation is found for the glacier bed. Let the glacier bed be given by the equation

$$F(x,y,z,t) := z_b(x,y,z,t) - z = 0.$$

Then it follows that

$$\boxed{\frac{\partial z_b}{\partial t} + v_x \frac{\partial z_b}{\partial x} + v_y \frac{\partial z_b}{\partial y} - v_z = -\dot{b}_b}, \tag{2.18}$$

where $\dot{b}_b$ is the ablation/accumulation rate at bed, which is usually small compared to the other terms.

2.7 Strain rates

We consider a line element $d\mathbf{x}^\circ$ of the medium K at $t = t^\circ$ (Fig. 2.8). The line element connects two material points of the medium. With time the medium will deform and therefore the length and the orientation of the line element will change. We will investigate the connection between the velocity field and the deformation of the line element.

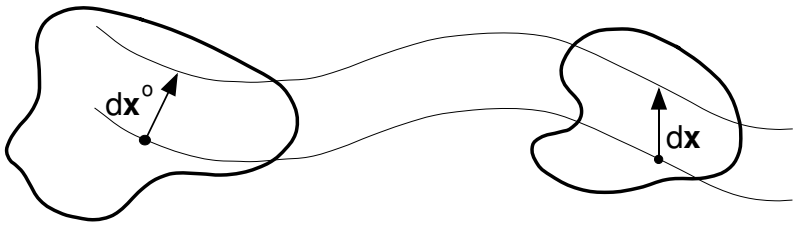

Figure 2.8: The movement of a line element $d\mathbf{x}^\circ$ through space.

The position of the material point $\mathbf{x}^\circ$ is given by

$$\mathbf{x} = \mathbf{x}(\mathbf{x}^\circ, t).$$

We consider an infinitesimal change in $\mathbf{x}$ such that

$$dx_i = \sum_{j=1}^{3} \underbrace{\frac{\partial x_i}{\partial x_j^\circ}}_{=:F_{ij}} dx_j^\circ,$$

which can also be written on the form

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{x}^\circ. \tag{2.19}$$

The elements

$$F_{ij} = \frac{\partial x_i}{\partial x_j^\circ} \tag{2.20}$$

of the tensor $\mathbf{F}$ are called the *deformation gradients* (German: *Deformationsgradient*).

Eq. (2.19) describes the transformation of the line element $d\mathbf{x}^\circ$ into $d\mathbf{x}$. Both $d\mathbf{x}^\circ$ and $d\mathbf{x}$ connect the same material points. In order to come up with a relation between the velocity field and the deformation of the line element we consider the material derivative of the deformation gradient

$$\begin{aligned}
\frac{DF_{ij}}{Dt} &= \frac{D}{Dt}\frac{\partial x_i}{\partial x_j^\circ} = \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial x_j^\circ}\right)\bigg|_{x^\circ} \\
&= \frac{\partial^2 x_i}{\partial x_j^\circ \partial t}\bigg|_{x^\circ} = \frac{\partial}{\partial x_j^\circ} v_i \\
&= \frac{\partial v_i}{\partial x_j^\circ} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial x_j^\circ}
\end{aligned} \tag{2.21}$$

which can also be written in the form

$$\boxed{\frac{D\mathbf{F}}{Dt} = (\nabla \mathbf{v})\mathbf{F}} \tag{2.22}$$

Exercise 36. *Show that Eq. (2.22) and Eq. (2.21) are identical.*

We now consider the material derivative of $d\mathbf{x}$

$$\begin{aligned}
\frac{D\, d\mathbf{x}}{Dt} &= \frac{D}{Dt}(\mathbf{F}\, d\mathbf{x}^\circ) \qquad \text{(Eq. (2.19))} \\
&= \frac{D\mathbf{F}}{Dt}\, d\mathbf{x}^\circ \qquad (D(d\mathbf{x}^\circ)/Dt = 0) \\
&= (\nabla \mathbf{v})\mathbf{F}\, d\mathbf{x}^\circ \qquad \text{(Eq. (2.22))} \\
&= (\nabla \mathbf{v})\, d\mathbf{x} \qquad \text{(Eq. (2.19))}.
\end{aligned}$$

or

$$\boxed{\frac{D}{Dt}(d\mathbf{x}) = (\nabla \mathbf{v})(d\mathbf{x})} \tag{2.23}$$

Both changes in the length and the orientation of the line element $d\mathbf{x}$ contribute to $Dd\mathbf{x}/Dt$. We would like to separate these two contributions, and therefore write

$$d\mathbf{x} = \underbrace{\frac{d\mathbf{x}}{|d\mathbf{x}|}}_{=:\mathbf{a}} \underbrace{|d\mathbf{x}|}_{=:ds},$$

or

$$d\mathbf{x} = \mathbf{a}\, ds. \tag{2.24}$$

The material derivative of $d\mathbf{x}$ can then be written as

$$\begin{aligned}
\frac{D}{Dt}(d\mathbf{x}) &= \frac{D}{Dt}(\mathbf{a}\, ds) \\
&= \frac{D(ds)}{Dt}\,\mathbf{a} + \frac{D\mathbf{a}}{Dt}\, ds.
\end{aligned} \tag{2.25}$$

From Eq. (2.23) and the fact that $(\nabla\mathbf{v})\,d\mathbf{x} = (\nabla\mathbf{v})\mathbf{a}\,ds = ds(\nabla\mathbf{v})\mathbf{a}$ it follows that

$$\begin{aligned}\frac{D}{Dt}(d\mathbf{x}) &= (\nabla\mathbf{v})(d\mathbf{x}) \\ &= ds(\nabla\mathbf{v})\mathbf{a}\end{aligned}$$

which allows us to write Eq. (2.25) on the form

$$ds\,(\nabla\mathbf{v})\mathbf{a} = \frac{D(ds)}{Dt}\mathbf{a} + ds\frac{D\mathbf{a}}{Dt}.$$

By multiplying both sides with $\mathbf{a}$ and dividing by ds we obtain

$$\mathbf{a}\cdot(\nabla\mathbf{v})\mathbf{a} = \frac{1}{ds}\frac{D(ds)}{Dt} + 0 \tag{2.26}$$

The second term on the right-hand side is equal to zero because

$$\begin{aligned}\mathbf{a}\cdot\frac{D\mathbf{a}}{Dt} &= \mathbf{a}\cdot\left(\frac{\partial\mathbf{a}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{a}\right) \\ &= \frac{1}{2}\frac{\partial(\mathbf{a}\cdot\mathbf{a})}{\partial t} + \mathbf{a}\cdot\left(v_x\frac{\partial a_x}{\partial x} + v_y\frac{\partial a_x}{\partial y} + v_z\frac{\partial a_x}{\partial z}, \cdots, \cdots\right) \\ &= \underbrace{\frac{\partial 1}{\partial t}}_{=0} + a_x v_x\frac{\partial a_x}{\partial x} + a_x v_y\frac{\partial a_x}{\partial y} + a_x v_z\frac{\partial a_x}{\partial z} + \cdots + \cdots \\ &= v_x\frac{1}{2}\frac{\partial a_x^2}{\partial x} + v_y\frac{1}{2}\frac{\partial a_x^2}{\partial y} + v_z\frac{1}{2}\frac{\partial a_x^2}{\partial z} + \cdots + \cdots \\ &= \frac{1}{2}\left(v_x\left(\frac{\partial a_x^2}{\partial x} + \frac{\partial a_y^2}{\partial x} + \frac{\partial a_z^2}{\partial x}\right) + v_y\left(\frac{\partial a_x^2}{\partial y} + \frac{\partial a_y^2}{\partial y} + \frac{\partial a_z^2}{\partial y}\right) + v_z\left(\frac{\partial a_x^2}{\partial z} + \frac{\partial a_y^2}{\partial z} + \frac{\partial a_z^2}{\partial z}\right)\right) \\ &= \frac{1}{2}\left(v_x\frac{\partial}{\partial x}\mathbf{a}\cdot\mathbf{a} + v_y\frac{\partial}{\partial y}\mathbf{a}\cdot\mathbf{a} + v_z\frac{\partial}{\partial z}\mathbf{a}\cdot\mathbf{a}\right) \\ &= \frac{1}{2}\left(v_x\frac{\partial}{\partial x}1 + v_y\frac{\partial}{\partial y}1 + v_z\frac{\partial}{\partial z}1\right) \\ &= 0.\end{aligned}$$

Question: Why is $\mathbf{a}\cdot\mathbf{a} = 1$?

We therefore have

$$\frac{1}{ds}\frac{D(ds)}{Dt} = \mathbf{a}\cdot(\nabla\mathbf{v})\mathbf{a}.$$

But since

$$\begin{aligned}\mathbf{a}\cdot(\nabla\mathbf{v})\mathbf{a} &= a_i v_{i,j} a_j \\ &= \frac{1}{2}a_i\left(v_{i,j} + v_{i,j}\right)a_j \\ &= \frac{1}{2}\left(a_i v_{i,j} a_j + a_i v_{i,j} a_j\right) \\ &= \frac{1}{2}\left(a_i v_{i,j} a_j + a_j v_{j,i} a_i\right) \qquad \text{(swap } i \text{ and } j \text{ in the second term)} \\ &= \frac{1}{2}\left(a_i v_{i,j} a_j + a_i v_{j,i} a_j\right) \qquad \text{(swap } a_i \text{ with } a_j \text{ in the second term)} \\ &= \frac{1}{2}a_i\left(v_{i,j} + v_{j,i}\right)a_j \\ &= \frac{1}{2}\mathbf{a}\cdot\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)\mathbf{a}\end{aligned}$$

we also have

$$\frac{1}{ds}\frac{D(ds)}{Dt} = \frac{1}{2}\mathbf{a}\cdot\underbrace{(\nabla\mathbf{v}+\nabla\mathbf{v}^{\mathbf{T}})}_{=:2\mathbf{D}}\mathbf{a}.$$

with the tensor D defined as

$$\mathbf{D} := \frac{1}{2}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^{\mathbf{T}}\right) \tag{2.27}$$

The tensor $\mathbf{D}$ is called *stretching tensor* or *the rate-of-deformation tensor* (German: *Verzerrungsgeschwindigkeitstensor*). We have arrived at

$$\boxed{\frac{1}{ds}\frac{D(ds)}{Dt} = \mathbf{a}\cdot\mathbf{D}\mathbf{a}} \tag{2.28}$$

Eq. (2.28) gives us the information which we were looking for. On the left-hand side we have the relative rate-of-change with time of the length of the line element $d\mathbf{x}^\circ$ as we follow the material particles that it connects. This rate is called the *stretching rate* at $\mathbf{x}$ along the direction $\mathbf{a}$. On the right-hand side we have the stretching tensor $\mathbf{D}$ and the vector $\mathbf{a}$ defining the orientation of the line element. The stretching tensor is completely determined by the spatial velocity gradient. If we know the velocity field, we can use the right-hand side of Eq. (2.28) to calculate the stretching rate $(ds^{-1}\,D(ds)/Dt)$ at any given point $\mathbf{x}$ of the medium in any given direction $\mathbf{a}$.

Question: Can the stretching rate be calculated by the right-hand side of Eq. (2.28) for any given velocity field, or only if the velocity gradients are small?

If follows from the definition 2.27 that $\mathbf{D}$ is a symmetrical tensor, that is

$$\begin{aligned}[\mathbf{D}]_{ij} &= [\mathbf{D}]_{ji}\\ [\mathbf{D}]_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\end{aligned}$$

The components of the stretching tensor are usually written as

$$\boxed{\dot{\epsilon}_{ij} := \frac{1}{2}(v_{i,j}+v_{j,i})} \tag{2.29}$$

and called *strain rates.*

Exercise 37. *Show that Eq. (2.28) can be written as*

$$\frac{D}{Dt}\ln(ds) = a_i a_j \dot{\epsilon}_{ij}$$

Exercise 38. *Calculate the stretching along the* $\hat{\mathbf{e}}_1$ *direction in terms of the strains rates* $\dot{\epsilon}_{ij}$*.*

Solution: We insert $\hat{\mathbf{e}}_1$ for $\mathbf{a}$ in Eq. (2.28):

$$\begin{aligned}\frac{1}{ds}\frac{D\,(ds)}{Dt} &= \hat{\mathbf{e}}_1\cdot\mathbf{D}\hat{\mathbf{e}}_1\\ &= \hat{\mathbf{e}}_1\cdot\begin{pmatrix}D_{11}\\ D_{21}\\ D_{31}\end{pmatrix}\\ &= D_{11}=\dot{\epsilon}_{11}\end{aligned}$$

This shows that $\dot{\epsilon}_{11}$ represents the stretching rate along $\hat{\mathbf{e}}_1$ direction. The components $\dot{\epsilon}_{11}$, $\dot{\epsilon}_{22}$ and $\dot{\epsilon}_{33}$, thus, are the stretching rates along $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ directions, respectively.

Exercise 39. *What is the geometrical meaning of the off-diagonal components of the stretching tensor?*

Chapter 3

Forces and stresses

3.1 Body forces and surface forces

All forces can be divided into *body* or *surface* forces (German: *Volumenlasten* and *Oberflächenlasten*). Body forces act within the body. The gravity force is an example for a body force. Surface forces act on the bounding surface of the body.

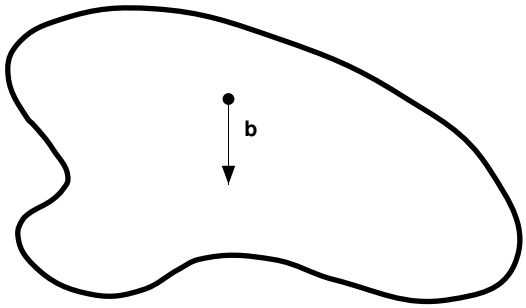

Figure 3.1: Body force.

Let the body force at the point $\mathbf{r}$ be $\mathbf{b}(\mathbf{r})$. The total body force $\mathbf{f}^V$, is the integral over the volume of the body

$$\mathbf{f}^V = \int_V \rho \mathbf{b}\, dV,$$

where ρ is the density. In a coordinate system with the $\hat{\mathbf{e}}_3$ axis pointing upward the gravity force is, for example,

$$\mathbf{b} = -g\,\hat{\mathbf{e}}_3,$$

with $g = 9.81\,\mathrm{m\,s^{-2}}$. The product $\rho\mathbf{b}$ has the units force per volume.

We consider the surface elements dA in Fig. 3.2. The orientation of this surface element is defined as the direction of the unit normal to the surface $\hat{\mathbf{n}}$. The surface force per unit of area $\mathbf{s}$ is the so called *surface traction* (German: *Spannungsvektor*). Other terms for the surface traction $\mathbf{s}$ are *stress vector*, and *surface force per unit of area*. The surface can be a part of the exterior of the body, or we can think of it as a part of an internal cut plane. If dA is a surface element in the interior of the body, we think of

$$\mathbf{s} = \mathbf{s}(\mathbf{r}, \hat{\mathbf{n}}, t)$$

as the internal surface force per unit area resulting from that part of the body into which $\hat{\mathbf{n}}$ is directed into.

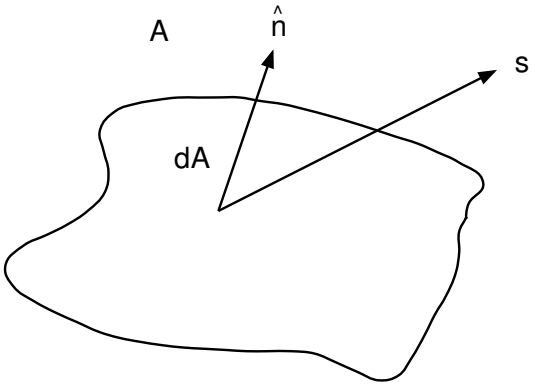

Figure 3.2: Surface force.

The surface traction is to be thought of as the limit

$$\mathbf{s} := \lim_{\triangle A \to 0} \frac{\triangle \mathbf{f}}{\triangle A}$$

where $\triangle \mathbf{f}$ is the surface force acting on the area $\triangle A$. It is important to realise that the surface traction will, in general, depend on both the location $\mathbf{r}$ and the orientation $\hat{\mathbf{e}}$ of the surface element dA, as well as on the time variable t, that is

$$\mathbf{s} = \mathbf{s}(\mathbf{r}, \hat{\mathbf{n}}, t).$$

The total surface force acting on a surface A is given by the integral

$$\mathbf{f}^A = \int_A \mathbf{s}\, dA.$$

Exercise 40. *Think about at least one example where the surface traction depends on the orientation of the surface.*

If we consider a surface element ΔA within a body, there are always two possible directions for $\hat{\mathbf{n}}$. If we arbitrarily chose one of these directions as $\hat{\mathbf{n}}$, then the surface traction

$$\mathbf{s}(\mathbf{r}, \hat{\mathbf{n}}, t)$$

is the surface force per unit area resulting from the part of the body to which $\hat{\mathbf{n}}$ is directed upon. The surface traction

$$\mathbf{s}(\mathbf{r}, -\hat{\mathbf{n}}, t)$$

is the surface force per unit area acting on ΔA from the other side. We can think about $\mathbf{s}(\mathbf{r}, \hat{\mathbf{n}}, t)$ as the action and the $\mathbf{s}(\mathbf{r}, -\hat{\mathbf{n}}, t)$ as the reaction surface force per unit area (or the other way around). From Newton's third law of motion we conclude that [1]

$$\begin{aligned} \mathbf{s}(-\hat{n}) &= -\mathbf{s}(\hat{n}) \\ (\text{actio} &= \text{reactio}) \end{aligned} \tag{3.1}$$

This relation is known as *Cauchy's lemma* or as *Cauchy's reciprocal relation*. Cauchy's lemma states that the force due to the 'outside' on the 'inside' is equal in magnitude and opposite to the force due to the 'inside' to the 'outside'.

[1] We will often simply write $\mathbf{s}(\hat{n})$ instead of the somewhat more accurate notation $\mathbf{s}(\mathbf{r}, \hat{n}, t)$.

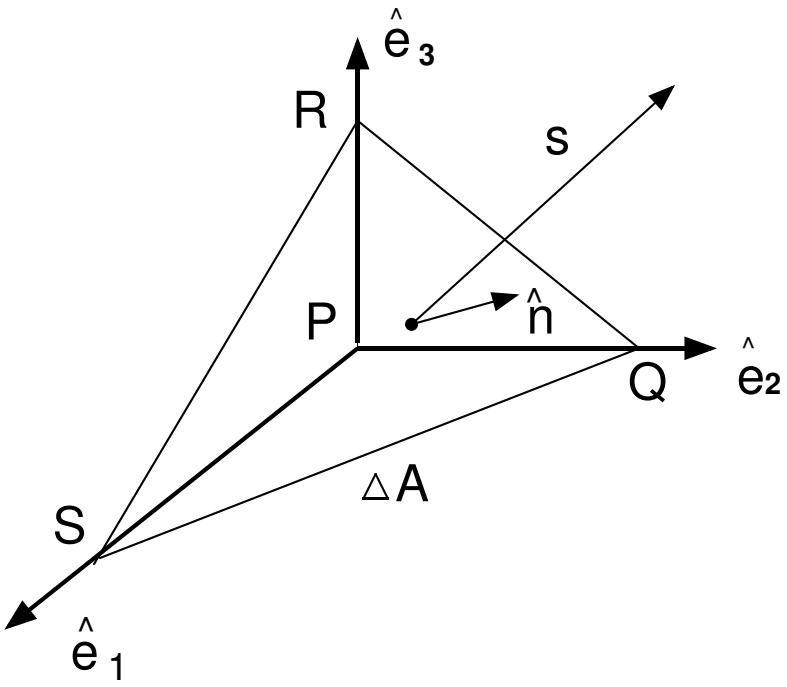

Figure 3.3: The slant face QSR is perpendicular to the exterior normal n. The vector $\mathbf{s}$ is the stress vector (traction) on that plane.

3.2 The stress tensor

The surface traction (which is as vector quantity) at some time t depends on both the location and the orientation of the surface element. We can never talk about the traction at some point $\mathbf{r}$ without specifying the orientation $\hat{\mathbf{n}}$. It turns out that it is possible to completely specify the traction (the stress vector) at a point for any given surface-element orientation $\hat{\mathbf{n}}$ if we know six quantities. These six quantities together form the components of a symmetrical tensor. To see this we consider the tetrahedron in Fig. 3.3.

The basic idea is to consider the relationship between the traction on planes perpendicular to the coordinate axes and on some plane with an arbitrary orientation. If such a relationship exists, then it is sufficient to specify the traction on the three planes perpendicular to the coordinate axes at every point, in order to calculate the traction on any plane at that point.

The slant face bounded by the points R, S, and Q in Fig. 3.3 has an arbitrary direction. It has the area $\triangle A$, and the traction on this area is $\mathbf{s}(\mathbf{n})$. Let $\triangle A_i$, for $i = 1, 2, 3$, be the areas of the faces of the tetrahedron perpendicular to the $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ axes, respectively. The triangles $\triangle A_i$ are, thus, the projections of the slant face $\triangle A$ on the three planes defined by $x_i = 0$, that is

$$\triangle A_i = \triangle A \, n_i \tag{3.2}$$

with

$$n_i = \hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i = \cos(\hat{n}_i, \hat{\mathbf{e}}_i).$$

By the second law of Newton the sum of all forces equals the product of mass and acceleration (assuming constant mass)

$$\underbrace{\text{Sum of all forces}}_{\mathbf{f}} = \underbrace{\text{mass}}_{\rho \triangle V} \cdot \underbrace{\text{acceleration}}_{\mathbf{a}} \tag{3.3}$$

We now from the sum of all forces in each direction of the coordinate system. The volume force in $\hat{\mathbf{e}}_i$ direction is

$$\mathbf{b} \cdot \hat{\mathbf{e}}_i = \rho \triangle V \mathbf{g} \cdot \hat{\mathbf{e}}_i.$$

The surface traction on, for example, the plane element with the orientation $\hat{\mathbf{e}}_2$ (bounded by the points PSR) is given by $-\mathbf{s}(\hat{\mathbf{e}}_2)$. The minus sign follows from the fact that the normal vector $\hat{\mathbf{e}}_2$ points *into* the tetrahedron. The *exterior* normal to the face PSR is $-\hat{\mathbf{n}}_2$. By Cauchy's Lemma $\mathbf{s}(-\hat{\mathbf{e}}_2) = -\mathbf{s}(\hat{\mathbf{e}}_2)$. The total force on that plane element is $\mathbf{s}(\hat{\mathbf{e}}_2)\triangle A_2$. The component of the total force along the $\hat{\mathbf{e}}_1$ direction is $\mathbf{s}(\hat{\mathbf{e}}_2)\triangle A_2\hat{\mathbf{e}}_1$. In general the surface force component of the $\triangle A_i$ surface in the $\hat{\mathbf{e}}_j$ direction is given by

$$\mathbf{s}(\hat{\mathbf{e}}_i)\triangle A_i \cdot \hat{\mathbf{e}}_j.$$

The surface force on the slant face QSR in $\hat{\mathbf{e}}_j$ direction is given by

$$\mathbf{s}(\hat{\mathbf{n}})\triangle A \cdot \hat{\mathbf{e}}_j.$$

We do not need the minus sign here because the normal vector $\hat{\mathbf{n}}$ points out of the tetrahedron.

For the $\hat{\mathbf{e}}_1$ direction Newton's second law of motion then gives

$$\triangle V\,\rho\mathbf{a}\cdot\hat{\mathbf{e}}_1 = -\triangle A_1\,\mathbf{s}(\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_1 - \triangle A_2\,\mathbf{s}(\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_1 - \triangle A_3\,\mathbf{s}(\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_1 + \mathbf{s}(\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}_1\triangle A + \rho\triangle V\mathbf{g}\cdot\hat{\mathbf{e}}_1 \quad (3.4)$$

On the right-hand side we have the total sum of volume and surface forces in the $\hat{\mathbf{e}}_1$ direction. If we now use

$$\triangle V = \frac{1}{3}h\triangle A$$

where h is the altitude of the tetrahedron[2] and Eq. (3.2) we can rewrite Eq. (3.4) as

$$\frac{1}{3}h\triangle A\,\rho\mathbf{a}\cdot\hat{\mathbf{e}}_1 = -\triangle A\,n_1\mathbf{s}(\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_1 - \triangle A\,n_2\mathbf{s}(\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_1 - \triangle A\,n_3\mathbf{s}(\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_1 + \mathbf{s}(\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}_1\triangle A + \rho\frac{1}{3}h\triangle A\mathbf{g}\cdot\hat{\mathbf{e}}_1 \quad (3.5)$$

Letting $h \to 0$ leads to

$$0 = -\triangle A\,n_1\mathbf{s}(\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_1 - \triangle A\,n_2\mathbf{s}(\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_1 - \triangle A\,n_3\mathbf{s}(\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_1 + \mathbf{s}(\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}_1\triangle A \quad (3.6)$$

or

$$0 = -n_1\mathbf{s}(\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_1 - n_2\mathbf{s}(\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_1 - n_3\mathbf{s}(\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_1 + \mathbf{s}(\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}_1 \quad (3.7)$$

As h approach zero, the volume and the four surface areas simultaneously go to zero and the tetrahedron shrinks to a point. The above expression is, thus, valid at the point P.

By introducing the notation

$$\sigma_{ij} := \mathbf{s}(\hat{\mathbf{e}}_i)\cdot\hat{\mathbf{e}}_j \quad (3.8)$$

we can write Eq. (3.7) in the form

$$\mathbf{s}(\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}_1 = \sigma_{i1}n_i.$$

Repeating the exercise to the other two directions shows that in general

$$\boxed{S_i(\hat{\mathbf{n}}) = \sigma_{ki}n_k} \quad (3.9)$$

Remark: We did not have to make any assumption about force balance in deriving (3.9). It remains true even if the body accelerates.

Eq. (3.9) expresses the relationship between the stress vectors on the faces $\triangle A_i$ and the stress vector $\mathbf{s}(\hat{\mathbf{n}})$ on the face $\triangle A$. The orientation of the surface element $\triangle A$ was completely arbitrary. We, thus, only have to know the components of the stress vectors $\mathbf{s}_j(\hat{\mathbf{e}}_i)$ in some coordinate system in order to calculate the stress vector (traction) $\mathbf{s}(\hat{\mathbf{n}})$.

The quantities σ_{ij} were defined above simply as short notation for the product of $\mathbf{s}(\hat{\mathbf{e}}_i)$ and $\hat{\mathbf{e}}_j$ (Eq. (3.8)). We now show that these nine quantities do indeed form the components of a second order Cartesian tensor.

[2] The altitude h of the tetrahedron is the distance from P toward the slant plane QSR along the direction $\hat{\mathbf{n}}$.

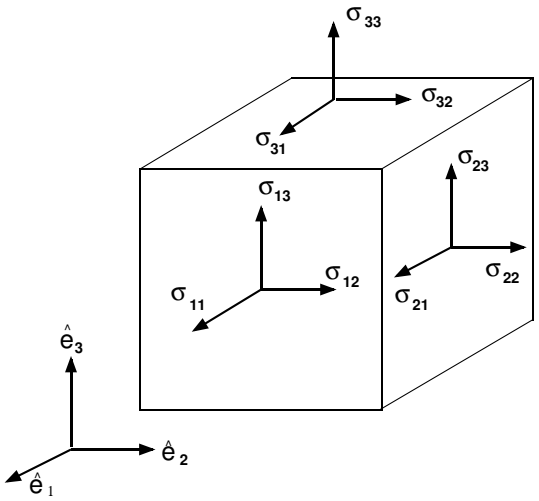

Figure 3.4: The physical interpretation of the individual components of the stress tensor follows directly from their definition (3.8). The first subscript relates to the orientation of the cut plane, the second to the component of the stress vector (traction) of the cut plane. The component σ_{ij} is, thus, the j component of the stress vector of the cut plane $\mathbf{s}(\hat{\mathbf{e}}_i)$, or simply $\sigma_{ij} = \mathbf{s}(\hat{\mathbf{e}}_i) \cdot \hat{\mathbf{e}}_j$, which is definition (3.8).

Exercise 41 (The quotient law)**.** *Proof that if the three components of the sum $a_{ij}b_i$ are components of a vector, with b_i being components of an arbitrary vector, then a_{ij} are components of a tensor.*

Solution. *In the coordinate system x_i let*

$$c_i = a_{ij}b_j$$

and in some other coordinate system x'_i let

$$c'_i = a'_{ij}b'_j.$$

Since c_i and b_j are both components of vectors, we have between c'_i, b'_i, and c_i, b_i, respectively, the usual rotational relationships between vectors ($c'_i = \alpha_{ij}c_j$ and $c_i = \alpha_{ji}c'_j$ where α_{ij} are the elements of the rotation matrix.).

$$\begin{aligned} a'_{ij}b'_j &= c'_i \\ &= \alpha_{iq}c_q \\ &= \alpha_{iq}a_{qp}b_p \\ &= \alpha_{iq}a_{qp}\alpha_{jp}\,b'_j \end{aligned}$$

or

$$0 = \left(\alpha_{iq}\alpha_{jp}\,a_{qp} - a'_{ij}\right) b'_j.$$

Since this is true for any vector with the components b'_i in system x'_i it follows that the term in the bracket must be equal to zero, that is

$$a'_{ij} = \alpha_{iq}\alpha_{jp}\,a_{qp}.$$

From the definition of a Cartesian tensor of second order it then follows that the quantities a_{ij} form the components of a tensor.

The quotient law shows that σ_{ij} are the components of a second order tensor tensor (Exercise: Explain this in detail). The tensor $\boldsymbol{\sigma}$ is called the *Cauchy's stress tensor*. (German: *Cauchyscher Spannungstensor* or *der Tensor der Reibungsspannungen*, and σ_{ij} *Reibungsspannungen*). The stress tensor describes the stress field completely.

The stress tensor is given by the tensor product

$$\boldsymbol{\sigma} = \sigma_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$$

Eq. (3.9) can then be written on the form

$$\mathbf{s}(\hat{\mathbf{n}}) = \boldsymbol{\sigma}^T \hat{\mathbf{n}} \tag{3.10}$$

or as

$$\mathbf{s}(\hat{\mathbf{n}}) = \begin{pmatrix} S_1(\hat{\mathbf{n}}) \\ S_2(\hat{\mathbf{n}}) \\ S_3(\hat{\mathbf{n}}) \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}^T \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \tag{3.11}$$

If we know the components of the stress tensor, Eq. (3.9) can be used to determine the traction (stress vector) on a plane with the unit normal $\hat{\mathbf{n}}$.

Exercise 42. *The components of the stress tensor are*

$$\boldsymbol{\sigma}^T = [\sigma_{ji}] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Find the traction on the plane defined by

$$x_1 + x_2 - 1 = 0$$

Also determine the angle θ between the stress vector $\mathbf{s}(\hat{\mathbf{n}})$ and $\hat{\mathbf{n}}$.

Solution. *The unit normal is*

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and from Eq. (3.11) we have

$$\begin{aligned} \mathbf{s}(\hat{\mathbf{n}}) &= \boldsymbol{\sigma}^T \hat{\mathbf{n}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \end{aligned}$$

The angle θ is

$$\begin{aligned} \cos\theta &= \frac{\mathbf{s}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}}{|\mathbf{s}(\hat{\mathbf{n}})|} = \frac{1}{\sqrt{2}} \frac{4}{\sqrt{26}} \\ \Rightarrow \quad \theta &= 56^\circ. \end{aligned}$$

3.2.1 The components of the stress tensor

A physical interpretation of individual terms can most easily be derived from the definition

$$\sigma_{ij} := \mathbf{s}(\hat{\mathbf{e}}_i) \cdot \hat{\mathbf{e}}_j.$$

This definition shows that σ_{ij} is the projection of the of the surface traction ($\mathbf{s}(\hat{\mathbf{e}}_i)$ along the unit normal vector $\hat{\mathbf{e}}_j$. Hence, σ_{ij} is the j component of the surface traction $\mathbf{s}(\hat{n}_i)$.

The traction $\mathbf{s}(\hat{\mathbf{n}})$ on the surface $F_{\hat{\mathbf{n}}}$ with the unit normal vector $\hat{\mathbf{n}}$ is the *external force* per unit area, that is the force that acts on $F_{\hat{\mathbf{n}}}$ by the adjacent part of the material into which $\hat{\mathbf{e}}$ is directed upon. The internal force, that is the force that is exerted on the surface by the part of the material that $\hat{\mathbf{e}}$ is pointing away from, is given by $\mathbf{s}(-\hat{\mathbf{n}})$. As mentioned above, there is a simple relationship between the external and the internal force known as the, Cauchy's reciprocal relation (see Eq. (3.1).

From the definition (3.8) we also see that σ_{ij} is a positive number if the absolute value of the angle between the surface traction $\mathbf{s}(\hat{\mathbf{e}}_i)$ and the orientation of the surface $\hat{\mathbf{e}}_j$ is less than 90°. Thus, for the area $F_{\hat{\mathbf{n}}_1}$ with the unit normal vector $\hat{\mathbf{n}}_1$, σ_{11} is positive if $\mathbf{s}(\hat{\mathbf{n}}_1)$ points 'out' of the area and negative if it points into it. Tensile stresses are therefore positive and compressive stresses negative.

3.2.2 Normal and shear stresses

As shown in Fig. 3.5 the traction on a surface can be written as a sum of a normal and a tangential vector, where

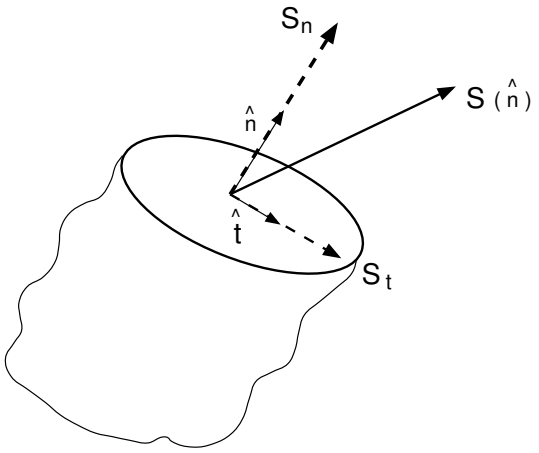

Figure 3.5: The traction ($\mathbf{s}(\hat{\mathbf{n}})$) and the corresponding normal ($\mathbf{s}_n$) and shear stresses ($\mathbf{s}_t$), with $\mathbf{s}(\hat{\mathbf{n}}) = \mathbf{s}_n + \mathbf{s}_t$.

$$\begin{aligned} S_n &= \hat{\mathbf{n}} \cdot \mathbf{s}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}^T \hat{\mathbf{n}} \\ S_t &= \hat{\mathbf{t}} \cdot \mathbf{s}(\hat{\mathbf{n}}) = \mathbf{s}(\hat{\mathbf{n}}) - S_n \hat{\mathbf{n}}. \end{aligned}$$

The quantity S_n is the normal stress and S_t is the tangential stress. Clearly one can never talk about normal or tangential stress without specifying ($\hat{\mathbf{n}}$) the orientation of the surface. However, somewhat confusingly, the main-diagonal elements of the stress tensor are also sometimes referred to as the normal stresses and the off diagonal elements as shear stresses.

3.2.3 Principal stresses

Consider a surface element with the normal vector $\hat{\mathbf{n}}$ and an associated stress vector $\mathbf{s}(\hat{\mathbf{n}})$. In general these two vectors will not be collinear. But if the stress tensor $\boldsymbol{\sigma}$ is given can we then find a surface with a a unit normal so that

$$\mathbf{s}(\hat{\mathbf{n}}) = \lambda \hat{\mathbf{n}}$$

where λ is a real scalar? If we write the above equation on the form

$$\mathbf{s}(\hat{\mathbf{n}}) = \boldsymbol{\sigma}^T \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}} \tag{3.12}$$

we see if we can solve this eigenvalue problem the eigenvector gives the orientation of the surface. As we will later see, the conservation of angular momentum implies that the Cauchy stress tensor $\boldsymbol{\sigma}$ must be symmetrical, that is

$$\sigma_{ij} = \sigma_{ji}.$$

Any textbook of linear algebra shows that for a symmetrical tensor the following statements hold:

- All eigenvalues are real and each eigenvector can be chosen so that each of its components is a real number.
- Eigenvectors corresponding to distinct eigenvalues are orthogonal.
- The mutually orthogonal eigenvectors can be used as a basis for a coordinate system. In this coordinate system all elements outside of the main diagonal are zero.

For the Cauchy stress tensor we can therefore always find a coordinate system in which the tensor is on a diagonal form

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}. \tag{3.13}$$

There is a bit of terminology used in this context. The three eigenvectors are called *principal axes*. The eigenvalues σ_1, σ_2, and σ_3, are the *principal stresses*. The principal stresses are real numbers, and the principal axes from a Cartesian coordinate system.

On page 14 we discussed the three principal invariants of a second-order tensor, and showed that the principal invariants of a second-order tensor are the coefficients of the characteristic equation. When written on the form (3.13), the characteristic equation of the Cauchy stress tensor is

$$\begin{aligned} \det(\boldsymbol{\sigma} - \lambda\mathbf{I}) &= \begin{vmatrix} \sigma_1 - \lambda & 0 & 0 \\ 0 & \sigma_2 - \lambda & 0 \\ 0 & 0 & \sigma_3 - \lambda \end{vmatrix} \\ &= (\sigma_1 - \lambda)(\sigma_2 - \lambda)(\sigma_3 - \lambda) \\ &= -\lambda^3 + \lambda^2(\sigma_1 + \sigma_2 + \sigma_3) - \lambda(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) - \sigma_1\sigma_2\sigma_3. \end{aligned}$$

Hence, in terms of the principal stresses the three fundamental stress invariants are

$$\begin{aligned} I_{\mathbf{A}} &= \sigma_1 + \sigma_2 + \sigma_3, \\ II_{\mathbf{A}} &= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3, \\ III_{\mathbf{A}} &= \sigma_1\sigma_2\sigma_3. \end{aligned}$$

3.2.4 Deviatoric stresses

As will become more clear in what follows it is often useful to decompose the Cauchy stress tensor into a spherical and a deviator part as follows

$$\boldsymbol{\sigma} = \underbrace{\frac{1}{3}\sigma_{ii}\mathbf{I}}_{\text{spherical part}} + \underbrace{\boldsymbol{\sigma}^{(d)}}_{\text{deviatoric part}}. \tag{3.14}$$

The deviatoric part is referred to as the *stress deviator* and its components as the *deviatoric stresses*.

We now define a quantity $\bar{p}$ as

$$\bar{p} := -\frac{1}{3}\sigma_{ii}. \tag{3.15}$$

The quantity $\bar{p}$ is simply the negative of the mean value of the elements along the main diagonal of the Cauchy stress tensor, i.e. the negative of the mean normal stress. Note that because the trace of the stress tensor is an invariant, $\bar{p}$ is also an invariant.

In the particular case of hydrostatic stress, $\bar{p}$ can be given a simple physical interpretation as follows. By definition stress at a given point $\mathbf{r}$ is hydrostatic if the same stress $-p$ acts on every area element going through that point, that is if

$$\mathbf{s}(\hat{\mathbf{n}}) = -p\,\hat{\mathbf{n}} \tag{3.16}$$

for any given area element with the orientation $\hat{\mathbf{n}}$. We can also write this equation on the form

$$\sigma\hat{\mathbf{n}} = -p\,\hat{\mathbf{n}} \tag{3.17}$$

which is an eigenvalue problem. It follows that $-p$ is the only eigenvalue and every unit vector an eigenvector. Every direction is a principal directions and the stress tensor is on a diagonal form in every Cartesian coordinate system. In this particular case the definition of $\bar{p}$ given above leads to

$$\bar{p} = -\frac{1}{3}\sigma_{ii} = -\frac{1}{3}(-p-p-p) = p.$$

Hence, for a hydrostatic stress field $\bar{p}$ is equal to the hydrostatic pressure. The quantity $\bar{p}$ is often referred to as the *mechanical pressure* or simply the *mean pressure*.

Pressure and incompressibility

In general $\bar{p}$ is not identical to the the thermodynamic pressure which in classical thermodynamics is defined through a equation of state ($p = p(T,\rho)$) relating the pressure to temperature (T) and specific density (ρ). For an incompressible media there is no such equation of state as, by definition, neither the density ρ nor the temperature T changes with pressure. In this case we simply define the pressure as being $\bar{p}$, write p instead of $\bar{p}$ and refer to p as the pressure.

For an incompressible material the pressure p is an independent variable. It does not enter the constitutive equations and, as we will see later, only its gradient enters the momentum equations. Unless it is specified somewhere as a part of the boundary conditions, the pressure can only be determined within an additive constant.

Stokes Hypothesis

If the material is compressible the two quantities $\bar{p}$ and p are distinct and not necessarily equal. To clarify this consider the general isotropic linear material equation

$$\sigma_{ij} = -p\delta_{ij} + \lambda\dot{\epsilon}_{kk}\delta_{ij} + 2\eta\dot{\epsilon}_{ij}$$

where p is the thermodynamical pressure and λ and η two material parameters. Using this equation together with the definition of $\bar{p}$ gives

$$\bar{p} = p - (\lambda + \frac{2}{3}\eta)\dot{\epsilon}_{ii}.$$

Thus, if $\dot{\epsilon}_{ii} = 0$ we have $\bar{p} = p$ showing that the variable $\bar{p}$ is a natural definition of pressure for incompressible medium. If, on the other hand, $\dot{\epsilon}_{ii} \neq 0$, then, in general $p \neq \bar{p}$ unless

$$\lambda = -\frac{2}{3}\eta. \tag{3.18}$$

G. G. Stokes suggested that this expression might hold for many materials, in which case $p = \bar{p}$. As it turns out this Stokes hypothesis, although not strictly correct for any materials, is sufficiently good for most incompressible flow situations. This Stokes hypothesis, however, continues to be a controversional subject. The quantity

$$\eta_D = \lambda + \frac{2}{3}\eta$$

is called the *bulk viscosity*. It is a measurable quantity and any deviation of its value from zero would indicate a violation of Stokes hypothesis. Direct measurements appear difficult to do and not many references are found in the literature. In basically all practical work it is assumed that $\eta_D \approx 0$ or at least $\eta_D \ll \eta$. Note that the Stokes hypothesis reduces the number of material parameters for an incompressible viscous material by one.

3.2.5 Invariants of the deviatoric stress tensor

The deviatoric stresses

$$\begin{aligned} \sigma_{ij}^{(d)} &= \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \\ &= \sigma_{ij} + p\,\delta ij \end{aligned}$$

describe the deviation of the stress field from a hydrostatic pressure field. For incompressible materials, only the deviatoric part of the stress tensor causes material deformation. In this sense the stress deviator is the 'interesting' part of the stress tensor.

There is a simple connection between the eigenvectors and the eigenvalues of the stress deviator with those of the stress tensor. If $\mathbf{s}$ is a eigenvector of the stress tensor $\boldsymbol{\sigma}$ for the eigenvalu λ. i.e.

$$\boldsymbol{\sigma}\mathbf{s} = \lambda\,\mathbf{s},$$

it follows that

$$\begin{aligned} \boldsymbol{\sigma}\,\mathbf{s} &= (\boldsymbol{\sigma}^{(d)} - p\mathbf{I})\,\mathbf{s} \\ &= \boldsymbol{\sigma}^{(d)}\mathbf{s} - p\,\mathbf{s} \\ &= \lambda\,\mathbf{s} \end{aligned}$$

and therfore

$$\boldsymbol{\sigma}^{(d)}\,\mathbf{s} = (\lambda + p)\,\mathbf{s}.$$

Thus, the stress deviator and the stress tensor have the same eigenvectors, and the eigenvalues are connected through

$$\lambda^{(d)} = \lambda + p.$$

The first invariant of the stress deviators is identically equal to zero:

$$\begin{aligned} I_{\boldsymbol{\sigma}}^{(d)} &= \sigma_{ii}^{(d)} \\ &= \sigma_{11}^{(d)} + \sigma_{22}^{(d)} + \sigma_{33}^{(d)} \\ &= \sigma_{11} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &+ \sigma_{22} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &+ \sigma_{33} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &= 0 \end{aligned}$$

The second invariant can be simplified to

$$\begin{aligned} \mathit{II}_{\boldsymbol{\sigma}}^{(d)} &= \frac{1}{2}(\underbrace{\sigma_{ii}^{(d)}}_{0}\underbrace{\sigma_{jj}^{(d)}}_{0} - \sigma_{ij}^{(d)}\sigma_{ji}^{(d)}) \\ &= -\frac{1}{2}\sigma_{ij}^{(d)}\sigma_{ji}^{(d)} \\ &= -\frac{1}{2}\mathrm{Sp}\left(\boldsymbol{\sigma}^{(d)}\boldsymbol{\sigma}^{(d)}\right). \end{aligned}$$

The third invariant is, as before, equal to the determinat of the stress deviator, or

$$III_{\boldsymbol{\sigma}}^{(d)} = \det(\sigma_{ij}^{(d)}).$$

If we know the eigenvalues of the stress deviator the invariants can be calculated in the same way as for any other symmetrical second-order tensor using

$$\begin{aligned} II_{\boldsymbol{\sigma}}^{(d)} &= \sigma_1^{(d)}\sigma_2^{(d)} + \sigma_2^{(d)}\sigma_3^{(d)} + \sigma_3^{(d)}\sigma_1^{(d)}, \\ III_{\boldsymbol{\sigma}}^{(d)} &= \sigma_1^{(d)}\sigma_2^{(d)}\sigma_3^{(d)}. \end{aligned}$$

where $\sigma_1^{(d)}$, $\sigma_2^{(d)}$, and $\sigma_3^{(d)}$ are the three eigenvalues.

Sometimes it is useful to know the second invariant of the stress deviator in terms of the eigenvalues of the stress tensor. Using the relationship between the eigenvalues of these two tensors given above we find after some algebra that

$$II_{\boldsymbol{\sigma}}^{(d)} = -\frac{1}{6}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right).$$

Exercise 43. *Show that*

$$\frac{\partial I_{\boldsymbol{\sigma}}}{\partial \sigma_{ij}} = \delta_{ij}, \tag{3.19}$$

and also that

$$\frac{\partial II_{\boldsymbol{\sigma}^{(d)}}}{\partial \sigma_{ij}} = -\sigma_{ij}^{(d)}. \tag{3.20}$$

Chapter 4

Conservation laws

The field variables (momentum field, energy field, etc.) are subjected to the basic fundamental conservation laws of nature. The mass, momentum, and energy of a closed system do, for example, not change with time. These conservation laws lead, within the context of continuum mechanics, to a set of field equations that hold at every point of a continuum and for all time. They relate various field variables and are independent of the material properties of the medium.

4.1 Integral formulas

A number of integral formulas are of great utility in deriving the conservation laws. We will state these formulas without a proof.

Consider the integral

$$I(t) = \int_{V(t)} T_{i_1 i_2 \ldots i_n}(x_i, t)\, dV.$$

The function $T_{i_1 i_2 \ldots i_n}(x_i, t)$ is a tensor of order n, and the integration follows over a time-dependent volume $V(t)$. The volume V encloses a body, and the surface of this body (A) is a material surface. The integral $I(t)$ is, thus, for all t evaluated over the same set of material points. The rate-of-change of $I(t)$ with time therefore corresponds to a material derivative.

4.1.1 Leibniz's Theorem

Leibniz's theorem is very useful for moving the (material) derivative inside an integral over a time-dependent volume. Let $A(t)$ be the surface of the volume $V(t)$, n_i the components of a unit normal vector of $A(t)$, and v_k the velocity components of a (material) particle. The Leibniz's theorem states that

$$\boxed{\frac{D}{Dt}\int_{V(t)} T_{i_1 i_2 \ldots i_n}(x_i, t)\, dV = \int_{V(t)} \frac{\partial T_{i_1 i_2 \ldots i_n}}{\partial t}\, dV + \int_{A(t)} n_k v_k T_{i_1 i_2 \ldots i_n}\, dA}. \tag{4.1}$$

The integral on the left-hand side is a material integral, that is the (material) volume V is the volume of a body enclosed by a material surface. Note that we *must* think of the integral on the left-hand side in this way because the time derivative (taken after the integration over $V(t)$) is a material derivative. For the first term on the right-hand side, on the other hand, this interpretation of the volume integral is not necessary. The result of this integral (first term on the right hand side) would not change if we were to substitute the material volume by a fixed integration range coinciding at time t with the material volume.

The theorem of Leibniz, therefore, gives us the relationship between 1) the rate-of-change with time of the integral of $T_{i_1 i_2 \ldots i_n}(x_i, t)$ over the material volume V moving with the body, 2) the integral of the local time derivative of $T_{i_1 i_2 \ldots i_n}(x_i, t)$ over a fixed region that at t coincides with

the time-dependent volume $V(t)$, and 3) the flux of $T_{i_1 i_2 \ldots i_n}(x_i, t)$ through the surface of that fixed region.

A useful special case of the theorem is obtained by setting $T_{i_1 i_2 \ldots i_n} = f(x,t)$.

$$\frac{D}{Dt}\int_{x=a(t)}^{x=b(t)} f(x,t)\,dx = \int_{x=a(t)}^{x=b(t)} \frac{\partial f(x,t)}{\partial t}\,dx + \frac{db(t)}{dt} f(b,t) - \frac{da(t)}{dt} f(a,t).$$

Exercise 44. *What about the special case $T_{i_1 i_2 \ldots i_n} = 1$? Can you interpret the resulting theorem?*

4.1.2 Reynold's transport formula

For $T_{i_1 i_2 \ldots i_n} = \phi(\mathbf{r}, \mathbf{t})$ we obtain the so called *Reynold's transport formula*

$$\boxed{\frac{D}{Dt}\int_V \phi\,dV = \int_V \frac{\partial \phi}{\partial t}\,dV + \int_A \phi\, n_k v_k\,dA}. \tag{4.2}$$

4.1.3 Integral formula of Gauss (divergence theorem)

The integral formula of Gauss is a further useful and well-known theorem

$$\int_V \frac{T_{i_1 i_2 \ldots i_n}}{\partial x_i}\,dV = \int_A n_i T_{i_1 i_2 \ldots i_n}\,dA. \tag{4.3}$$

The special case $T_{i_1 i_2 \ldots i_n} = \xi_i$, where ξ_i is a vector field leads to

$$\int_V \xi_{i,i}\,dV = \int_A n_i \xi_i\,dA. \tag{4.4}$$

4.2 Conservation of mass

The total mass of a material body having the volume $V(t)$ is given by the integral of density field $\rho(\mathbf{r}, t)$ over the total volume of the body

$$M = \int_{V(t)} \rho(\mathbf{r}, t)\,dV.$$

The conservation of mass reads

$$\frac{DM}{Dt} = 0.$$

Using the transport formula (with $\phi = \rho$ in Eq. (4.2)) and the integral formula of Gauss (with $\xi = v_i \rho$ in Eq. (4.4)) leads to

$$\begin{aligned} \frac{DM}{Dt} = \frac{D}{Dt}\int_{V(t)} \rho\,dV &= \int_{V(t)} \frac{\partial \rho}{\partial t}\,dV + \int_{A(t)} \rho n_k v_k\,dA \\ &= \int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(v_k \rho) \right) dV. \end{aligned} \tag{4.5}$$

Equation (4.5) holds for any volume at all times. It, thus, follows that

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(v_k \rho) = 0}, \tag{4.6}$$

or in vector notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{4.7}$$

Eq. (4.6) is usually referred to as the *equation of continuity* or the *continuity equation*. In glaciology the term 'continuity equation' is usually used for a different equation, and for that reason we will refer to Eq. (4.6) as the mass-conservation equation (German: *Massenerhaltungsgleichung*). The vector form of the mass-conservation equation (Eq. (4.7)) is independent of the particular coordinate system used.

Another useful form of mass-conservation equation is found by writing the second term on the left-hand side of Eq. (4.6) as

$$(\rho v_k)_{,k} = \rho v_{k,k} + v_k \rho_{,k} \,, \tag{4.8}$$

which then leads to

$$\underbrace{\underbrace{\frac{\partial \rho}{\partial t}}_{\frac{D\rho}{Dt}} + v_k \rho_{,k}}_{\frac{D\rho}{Dt} + \rho\, v_{k,k} = 0} + \rho v_{k,k} = 0, \tag{4.9}$$

or in vector notation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{4.10}$$

4.2.1 Incompressible continuum

In an incompressible continuum (German: dichtebeständiges Medium)[1] the density ρ remains unchanged during the motion. The rate-of-change of ρ as the motion of a material particle is followed is therefore zero

$$\frac{D\rho}{Dt} = 0 \quad \text{(incompressible continuum (dichtebeständiges Medium))}. \tag{4.12}$$

For an incompressible continuum it follows from Eq. (4.10) that

$$\nabla \cdot \mathbf{v} = 0,$$

or

$$\boxed{v_{k,k} = 0}. \tag{4.13}$$

This is known as the *condition of incompressibility*.

4.3 Another useful integral formula

The mass-conservation equation (4.6) can be used together with the Reynold's transport formula (4.2) to arrive at another useful integral formula.

Let ϕ be some scalar variable. The formula of Gauss (4.3) when applied for the product ϕv_i gives

$$\int_V (\phi v_i)_{,i} \, dV = \int_A \phi v_i n_i, \; dA.$$

The term on the right-hand side is identical to the second term on the right-hand side of the transport formula (4.2). The transport formula (4.2) can therefore be written on the alternative form

$$\frac{D}{Dt} \int_V \phi \, dV = \int_V \frac{\partial \phi}{\partial t} \, dV + \int_V (\phi v_i)_{,i} \, dV \tag{4.14}$$

[1] In German one uses the term **inkompressibel** if the density is independent of pressure, that is if

$$\frac{\partial \rho}{\partial p} = 0 \quad \text{(incompressible continuum)}. \tag{4.11}$$

Now using the identity

$$\frac{\partial \phi}{\partial t} + (\phi v)_{,i} = \frac{D\phi}{Dt} + \phi v_{,i}$$

Eq. (4.14) can written as

$$\frac{D}{Dt}\int_V \phi \, dV = \int_V \left(\frac{D\phi}{\partial t} + \phi v_{,i}) \right). \tag{4.15}$$

For $\phi = \rho\theta$ where ρ is the mass density and θ some arbitrary function (4.15) gives

$$\begin{aligned} \frac{D}{Dt}\int_V \rho\theta \, dV &= \int_V \left\{ \frac{D}{Dt}(\rho\theta) + \rho\theta v_{i,i} \right\} dV \\ &= \int_V \left\{ \rho \frac{D\theta}{Dt} + \theta \underbrace{\left(\frac{D\rho}{Dt} + \rho v_{i,i} \right)}_{=0} \right\} dV. \end{aligned}$$

The bracket on the right-hand side is zero because of the mass-conservation equation. We, therefore, have the useful relation

$$\boxed{\frac{D}{Dt}\int_V \theta\rho \, dV = \int_V \frac{D\theta}{Dt}\rho \, dV}. \tag{4.16}$$

In Eq. (4.16) the variable ρ stands for the density of the continuum. This fact was used in the derivation above (where?). The variable θ, on the other hand, is any field variable and may be a scalar, vector, or tensor function.

4.4 Balance of linear momentum

The sum of all forces acting on a body is the sum of all surface and body forces

$$\text{sum of all forces} = \underbrace{\int_V \rho b_i \, dV}_{\text{body force}} + \underbrace{\int_A s_i \, dA}_{\text{surface force}} \quad ,$$

where s_i are the components of the stress vector ($\mathbf{s}(\mathbf{r}, \hat{\mathbf{n}})$). The Newton's second law of motion is

$$\text{sum of all forces} = \underbrace{\text{rate-of-change of linear momentum}}_{\frac{D}{Dt}\int_V \rho \mathbf{v} \, dV}.$$

We, therefore, have

$$\frac{D}{Dt}\int_V \rho v_i \, dV = \int_V \rho b_i \, dV + \int_A s_i \, dA. \tag{4.17}$$

This equation expresses the conservation of linear momentum for a continuum. We now proceed to derive the local form of this expression.

Using (4.16) with $\theta = v_i$ on the left-hand side gives

$$\frac{D}{Dt}\int_V \rho v_i \, dV = \int_V \rho \frac{Dv_i}{Dt} \, dV \tag{4.18}$$

The surface integral in (4.17) can be written as a volume integral using (4.3) (with $T_{i_1 i_2 \ldots i_n} = \sigma_{ij}$)

$$\begin{aligned} \int_A s_i \, dA &= \int_A \sigma_{ki} n_k \, dA \\ &= \int_V \sigma_{ki,k} \, dV. \end{aligned} \tag{4.19}$$

Inserting (4.19) and (4.18) into (4.17) gives

$$\int_V \rho \frac{Dv_i}{Dt}\, dV = \int_V (\rho b_i + \sigma_{ki,k})\, dV.$$

This must be true for every volume V, and therefore the integrand must be equal to zero, or

$$\boxed{\rho \frac{Dv_i}{Dt} = \rho b_i + \sigma_{ki,k}}, \tag{4.20}$$

which in vector notation reads

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{b} + \nabla \cdot \sigma^T.$$

Equation (4.20) expresses the balance of linear momentum in a local form. It is usually called the *momentum equation* and is also known as *Cauchy's equations of motion* and as *Cauchy's first law of motion*. (German: *differentielle Impulsgleichung* or *Bewegungsgleichung*)

4.4.1 Stokes flow

For glaciers the acceleration term (left left-hand side of (4.20)) is much smaller than the other terms of that equation. A simple order-of-magnitude estimation is sufficient to see this. Flow velocities of glaciers are on the order of $100\,\mathrm{m a^{-1}}$, and the volume force (gravity) is $\rho g = 9.81 \times 917\,\mathrm{m\,s^{-2}} \approx 10^{19}\,\mathrm{m\,a^{-2}}$. The vertical stress gradient is on the order of $\partial(\rho g z)/\partial z = \rho g \approx 10^{19}\,\mathrm{m\,a^{-2}}$.

$$\underbrace{\frac{\partial v_i}{\partial t}}_{\sim 100\,\mathrm{m/a^2}} + \underbrace{\underbrace{v_k v_{i,k}}_{\sim 100\,\mathrm{m/a} \cdot \frac{100\,\mathrm{m/a}}{100\,\mathrm{m}}}}_{\sim 100\,\mathrm{m/a^2}} = \underbrace{b_i}_{\sim 10^{16}\,\mathrm{m/a^2}} + \underbrace{\sigma_{ki,k}\frac{1}{\rho}}_{\sim 10^{16}\,\mathrm{m/a^2}} \tag{4.21}$$

The acceleration term is, thus, about 10^{14} times smaller than the other terms, so that the momentum equation can be used on the form

$$\sigma_{ki,k} + \rho\, b_i = 0. \tag{4.22}$$

The acceleration term, which is only non-linear term of the momentum equation can, thus, be ignored.

In glaciology the Eq. (4.22) is usually referred to as the momentum equation. In continuum mechanics it is more common to refer to this equation as the *equilibrium equation*. Another often used term for Eq. (4.22) is *Stokes equation*.

In terms of the deviatoric stresses

$$\sigma_{ij} = \sigma^{(d)}_{ij} - p\,\delta_{ij}$$

the Stokes equation can be written as

$$\sigma^{(d)}_{ki,k} + \rho b_i = p_{,i}. \tag{4.23}$$

Exercise 45. *Show that* $(p\delta_{ki})_{,k} = p_{,i}$.

The motion of a fluid for which the acceleration terms are so small that they can be ignored is known as *Stokes flow*. Such type of motion is also called *creeping flow*.[2]

Note that the time variable does not enter the Stokes equation (4.22). Any time dependency arrives from the boundary conditions.

[2]The terms Stokes flow and *Stokesian fluid* have different meanings. For a Stokesian fluid the deviatoric stress tensor is a homogeneous function of the stretching tensor $\mathbf{D}$.

4.5 Balance of angular momentum

The law of balance of angular momentum states that the rate-of-change of the angular momentum equals the vector sum of all moments of external forces.

The angular momentum of a continuum about the origin is given by the integral

$$\begin{aligned} \mathbf{L} &= \int_V \mathbf{x} \times (\rho \mathbf{v})\, dV \\ \text{or} \quad L_i &= \int_V \rho\, \varepsilon_{ijk} x_j v_k\, dV \end{aligned}$$

The resultant moment of all volume forces is

$$\int_V \mathbf{x} \times (\rho \mathbf{b})\, dV \quad \text{or} \quad \int_V \rho\, \varepsilon_{ijk} x_j b_k\, dV,$$

and the resultant moment of all surface forces is

$$\int_A \mathbf{x} \times \mathbf{S}(\mathbf{x}, \hat{\mathbf{n}}, t)\, dA \quad \text{or} \quad \int_A \varepsilon_{ijk} x_j S_k\, dA.$$

The integral form of the law of balance of angular momentum is then

$$\underbrace{\frac{D}{Dt}\int_V \rho\, \varepsilon_{ijk} x_j v_k\, dV}_{\substack{\text{rate of change} \\ \text{of momentum}}} = \underbrace{\int_V \rho\, \varepsilon_{ijk} x_j b_k\, dV}_{\substack{\text{resulting moment} \\ \text{of volume forces}}} + \underbrace{\int_A \varepsilon_{ijk} x_j S_k\, dA}_{\substack{\text{resulting moment} \\ \text{of surface forces}}} . \tag{4.24}$$

We will now derive the local form of (4.24). We do this by 1) changing the order of differentiation and integration on the left-hand side of (4.24), and 2) by transferring the surface integral on the right-hand side into an integral over the volume V.

First we consider the term on the left-hand side of (4.24). In order to change the order of differentiation and integration we apply the integral formula (4.16) with $\theta = \varepsilon_{ijk} x_j v_k$ and arrive at

$$\frac{D}{Dt}\int_V \rho\, \varepsilon_{ijk} x_j v_k\, dV = \int_V \rho \varepsilon_{ijk} \frac{D}{Dt}\left(x_j v_k\right) dV. \tag{4.25}$$

We can also derive the above equation directly as follows:

$$\begin{aligned} &\frac{D}{Dt}\int_V \rho\, \varepsilon_{ijk} x_j v_k\, dV \\ &= \int_V \frac{\partial}{\partial t}\left(\rho \varepsilon_{ijk} x_j v_k\right) dV + \int_A n_q v_q \rho \varepsilon_{ijk} v_k x_j\, dV \quad \text{(Leibniz Theorem (4.1))} \\ &= \int_V \frac{\partial}{\partial t}\left(\rho \varepsilon_{ijk} x_j v_k\right) dV + \int_V \left(v_q \rho \varepsilon_{ijk} x_j v_k\right)_{,q} dV \quad \text{(Gauss integral formula (4.3))} \\ &= \int_V \Big\{ \varepsilon_{ijk} x_j v_k \frac{\partial \rho}{\partial t} + \rho \underbrace{\frac{\partial x_j}{\partial t}}_{v_j} v_k \varepsilon_{ijk} + \rho \varepsilon_{ijk} x_j \frac{\partial v_k}{\partial t} + (\rho v_q)_{,q} \varepsilon_{ijk} x_j v_k + v_q \rho \varepsilon_{ijk} \left(x_j v_k\right)_{,q} \Big\} dV \\ &= \int_V \Big\{ \varepsilon_{ijk} x_j v_k \underbrace{\left(\frac{\partial \rho}{\partial t} + (\rho v_q)_{,q}\right)}_{=0 \text{(mass conservation)}} + \underbrace{\rho \varepsilon_{ijk} v_j v_k}_{=[\mathbf{v}\times\mathbf{v}]_i = 0} + \rho \varepsilon_{ijk} x_j \frac{\partial v_k}{\partial t} + v_q \rho \varepsilon_{ijk} (x_j v_k)_{,q} \Big\} dV \\ &= \int_V \rho \varepsilon_{ijk} \left\{ \frac{\partial}{\partial t}(x_j v_k) + v_q \frac{\partial}{\partial x_q}\left(x_j v_k\right) \right\} \\ &= \int_V \rho \varepsilon_{ijk} \frac{D(x_j v_k)}{Dt}\, dV \end{aligned}$$

which is (4.25).

We now consider the second term on the right-hand side of (4.24):

$$\begin{aligned}\int_A \varepsilon_{ijk} x_j S_k \, dA &= \int_A \varepsilon_{ijk} x_j \sigma_{qk} n_q \, dA \quad \text{(Eq. (3.9))} \\ &= \int_V \varepsilon_{ijk} \frac{\partial}{\partial x_q} \left(x_j \sigma_{qk} \right) \, dV \quad \text{(Gauss integral formula (4.3))} \\ &= \int_V \left(\varepsilon_{ijk} \delta_{jq} \sigma_{qk} + x_j \sigma_{qk,q} \right) \, dV \quad \text{(Because } x_{j,q} = \delta_{jq}\text{)} \\ &= \int_V \varepsilon_{ijk} \left(\sigma_{jk} + x_j \sigma_{qk,q} \right) \, dV\end{aligned}$$

Putting this expression together with Eq. (4.25) into Eq. (4.24) gives the local form of the angular-momentum equation

$$\rho \varepsilon_{ijk} \frac{D(x_j v_k)}{Dt} = \rho \, \varepsilon_{ijk} x_j b_k + \varepsilon_{ijk} \sigma_{jk} + \varepsilon_{ijk} x_j \sigma_{qk,q}. \tag{4.26}$$

because the volume B is arbitrary.

Equation (4.26) can be simplified considerably. By rearranging terms of Eq. (4.26)

$$\varepsilon_{ijk} x_j \underbrace{\left(\rho \frac{Dv_k}{Dt} - \sigma_{qk,q} - \rho b_k \right)}_{=0 \quad \text{Eq. (4.20)}} + \underbrace{\rho \, \varepsilon_{ijk} v_k \underbrace{\frac{Dx_j}{Dt}}_{v_j}}_{=[\mathbf{v} \times \mathbf{v}]_i = 0} = \varepsilon_{ijk} \, \sigma_{jk}$$

we find that

$$\varepsilon_{ijk} \, \sigma_{jk} = 0. \tag{4.27}$$

By forming the triple sum $\varepsilon_{ijk} \varepsilon_{kpq} \sigma_{pk}$ and using the δ-ε-relation

$$\varepsilon_{ijk} \varepsilon_{kpq} = \delta_{ip} \delta_{jq} - \delta_{jp} \delta_{iq},$$

we obtain

$$\begin{aligned} 0 &= \varepsilon_{kpq} \sigma_{pq} \\ &= \varepsilon_{ijk} \varepsilon_{kpq} \sigma_{pq} \\ &= \left(\delta_{ip} \delta_{jq} - \delta_{jp} \delta_{iq} \right) \sigma_{pq} \\ &= \sigma_{ij} - \sigma_{ji}. \end{aligned}$$

or

$$\boxed{\sigma_{ij} - \sigma_{ji} = 0}. \tag{4.28}$$

The balance of angular momentum leads to the conclusion that the stress tensor is symmetric. Equations (4.28) are called Cauchy's second law of motion.

4.6 Energiebilanz

Nach den Gesetzen der Thermodynamik ist die zeitliche Änderung der gesamten Energie eines geschlossenes Systems gleich der Leistung der äusseren Kräfte plus der pro Zeiteinheit von aussen zugeführten Wärme. Wenn wir die gesamte Energie in die kinematische (K) und die innere Energie (E) aufteilen, können wir diese Aussage folgendermassen in Symbolen ausdrücken

$$\frac{D}{Dt}(K + E) = P + \dot{Q},$$

wobei P die gesamte von äusseren Kräften pro Zeiteinheit geleistete Arbeit darstellt, und $\dot{Q}$ die gesamte dem Körper pro Zeiteinheit zugeführte Wärmemenge ist. Diese Aussage bezieht sich auf ein geschlossenes System, wie z.B. einen materiellen Körper. Im wesentlichen wird durch diese Aussage postuliert, dass 1) unterschiedliche Energieformen, wie z.B. mechanische und thermische Energien, ineinander umgewandelt werden können, und 2), dass Energie dabei nicht vernichtet werden kann.

Table 4.1: Neue in diesem Abschnitt eingeführte Symbole

Symbol	Bedeutung
E	Gesamte innere Energie eines Körpers
K	Gesamte kinetische Energie eines Körpers
P	Die gesamte von äusseren Kräften pro Zeiteinheit geleistete Arbeit
$\dot{Q}$	Die gesamte dem Körper pro Zeiteinheit zugeführte Wärmemenge
e	innere Energie pro Masseneinheit
q_i	Komponenten des Wärmestromvektors
h	Wärmeproduktion pro Zeit- und Masseneinheit

Die gesamte innere Energie (E) eines Körpers ist eine extensive Grösse. Die spezifische innere Energie (innere Energie pro Masseneinheit) bezeichnen wir mit e. Die innere Energie eines materiellen Teilchens ist $e\rho\, dV$, und die gesamte innere Energie damit

$$E = \int_V \rho e \, dV,$$

wobei V das Volumen eines materiellen Körper darstellt. Dieses Volumen kann sich im allgemeinen mit der Zeit ändern.

Die kinetische Energie eines materiellen Teilchens ist $\rho v_i v_i/2\, dV$, und die gesamte kinetische Energie ist durch

$$K = \int_V \frac{1}{2}\rho v_i v_i \, dV$$

gegeben.

Die äusseren Kräfte teilen wir in Oberflächenkräfte und Volumenkräfte auf. Die gesamte geleistete Arbeit pro Zeiteinheit (die Leistung) setzt sich demnach aus der Leistung der Oberflächenkräfte an jedem Flächenelement dA der Oberfläche des Körpers ($s_i v_i\, dA$) und der Leistung der Volumenkräfte zusammen

$$P = \int_A s_i v_i \, dA + \int_V \rho v_i b_i \, dV.$$

Mit

$$s_i = \sigma_{ji} n_j \qquad \text{(Siehe Gl. (3.9))}$$

erhalten wir

$$P = \int_A \sigma_{ji} n_j v_i \, dA + \int_V \rho v_i b_i \, dV.$$

Die dem Körper zugeführte Wärmemenge setzt sich aus der im Körper produzierten Wärme (Radioaktiver Zerfall, Strahlung) und dem durch die Oberfläche tretenden Wärmefluss zusammen. Die pro Zeit- und Masseneinheit produzierte Wärme bezeichnen wir mit h, und den Wärmestrom durch ein Element der Oberfläche mit $-q_i n_i\, dA$. Der Vektor $\mathbf{q}$ ist der Wärmestromvektor. Das negative Vorzeichen sorgt dafür, dass einfliessende Energie als positive Grösse gerechnet wird

$$\dot{Q} = \int_V \rho h \, dV - \int_A q_i \, n_i \, dA.$$

Die Energiebilanz eines materiellen Körpers lautet damit

$$\frac{D}{Dt}\int_V \left(\frac{1}{2}v_i v_i + e\right)\rho\, dV = \int_V \rho v_i b_i\, dV + \int_A \sigma_{ji} n_j v_i\, dA + \int_V \rho h\, dV - \int_A q_i n_i\, dA. \tag{4.29}$$

4.6.1 Herleitung der lokalen Form der Energieerhaltung

Um eine lokale (differentielle) Form der Energiebilanz zu erhalten, vertauschen wir zuerst die Reihenfolge der materiellen Ableitung und des Volumenintegrals auf der linken Seite, und wandeln danach alle Flächenintegrale in entsprechende Volumenintegrale um.

Die Reihenfolge der materiellen Ableitung und des Volumenintegrals auf der rechten Seite von Gl. (4.29) tauschen wir nach dem Theorem (4.16) wie folgt um

$$\begin{aligned}\frac{D}{Dt}\int_V \left(\frac{1}{2}v_i v_i + e\right)\rho\, dV &= \int_V \left\{\frac{D}{Dt}\left(\left(\frac{1}{2}v_i v_i + e\right)\rho\right) + \left(\frac{1}{2}v_i v_i + e\right)\rho v_{k,k}\right\} dV \quad \text{(Gl. (4.16))}\\ &= \int_V \left\{\rho\frac{D}{Dt}\left(\frac{1}{2}v_i v_i + e\right) + \left(\frac{1}{2}v_i v_i + e\right)\left(\frac{D\rho}{Dt} + \rho v_{k,k}\right)\right\} dV.\end{aligned}$$

Aufgrund der Massenerhaltung ist die letzte Klammer Null (siehe Gl. (4.10)). Also ist

$$\begin{aligned}\frac{D}{Dt}\int_V \left(\frac{1}{2}v_i v_i + e\right)\rho\, dV &= \int_V \rho\frac{D}{Dt}\left(\frac{1}{2}v_i v_i + e\right) dV\\ &= \int_V \rho\left(v_i\frac{Dv_i}{Dt} + \frac{De}{Dt}\right) dV.\end{aligned} \tag{4.30}$$

Die beiden Flächenintegrale auf der rechten Seite von Gl. (4.29) lassen sich nach dem Gauss'schen Satz in entsprechende Volumenintegrale umwandeln, d.h.

$$\begin{aligned}\int_A \sigma_{ji} n_j v_i\, dA &= \int_V (\sigma_{ji} v_i)_{,j}\, dV \quad \text{(Gauss, Gl. (4.4))}\\ &= \int_V (\sigma_{ji,j} v_i + \sigma_{ji} v_{i,j})\, dV\end{aligned} \tag{4.31}$$

und

$$\int_A q_i n_i\, dA = \int_V q_{i,i}\, dV. \quad \text{(Gauss, Gl. (4.4))} \tag{4.32}$$

Nach diesen Umformungen einiger Integrale in der Gl. (4.29) können wir diese Gleichung jetzt wie folgt schreiben

$$\int_V \rho\left(v_i\frac{Dv_i}{Dt} + \frac{De}{Dt}\right) dV = \int_V \left(\rho v_i b_i + \sigma_{ji,j} v_i + \sigma_{ji} v_{i,j} + \rho h + q_{i,i}\right) dV.$$

Dieser Ausdruck lässt sich noch vereinfachen. Dazu schreiben wir zuerst anstelle der materiellen Ableitung die Summe der lokalen und der konvektiven Ableitung ($D/Dt = \partial/\partial t + v_j \partial/\partial x_j$) und erhalten

$$\int_V \left\{\left(\frac{\partial v_i}{\partial t} + v_j v_{i,j} - \rho b_i - \sigma_{ij,j}\right) v_i + \rho\frac{\partial e}{\partial t} + \rho v_i e_{,i} - \sigma_{ij} v_{i,j} + q_{i,i} - \rho h\right\} dV = 0.$$

Aufgrund der Impulserhaltung verschwindet der erste Klammerausdruck (Siehe Gl. (4.20)).

Die zweifache Summe $\sigma_{ij} v_{i,j}$ kann, da der Spannungstensor symmetrisch ist, folgendermassen

umgewandelt werden

$$\begin{aligned}\sigma_{ij}v_{i,j} &= \frac{1}{2}(\sigma_{ij}v_{i,j}+\sigma_{ij}v_{i,j})\\ &= \frac{1}{2}(\sigma_{ij}v_{i,j}+\sigma_{ji}v_{j,i})\\ &= \frac{1}{2}(\sigma_{ij}v_{i,j}+\sigma_{ij}v_{j,i})\\ &= \sigma_{ij}\frac{1}{2}(v_{i,j}+v_{j,i})\\ &= \sigma_{ij}\dot{\epsilon}_{ij}.\end{aligned}$$

Damit ist

$$\int_V \left\{\rho\frac{\partial e}{\partial t}+\rho v_i e_{,i}-\sigma_{ij}\dot{\epsilon}_{ij}+q_{i,i}-\rho h\right\}dV=0.$$

Da der Integrationsbereich beliebig ist und der Integrand als stetig vorausgesetzt wird, muss der Integrand verschwinden, d.h.

$$\boxed{\rho\frac{\partial e}{\partial t}+\rho v_i e_{,i}=\sigma_{ij}\dot{\epsilon}_{ij}-q_{i,i}+\rho h}. \tag{4.33}$$

Es ist interessant, diesen Ausdruck mit der Formulierung des ersten Hauptsatzes der klassischen Thermodynamik zu vergleichen. Nach dem ersten Hauptsatz gilt

$$de=\delta w+\delta q.$$

Hier ist de die Änderung der spezifischen inneren Energie in der Zeit dt, δw die in dieser Zeit verrichtete Arbeit, und δq die in dieser Zeit zugeführte Wärme. Ebenfalls gilt

$$\frac{de}{dt}=\delta\dot{w}+\delta\dot{q}.$$

In einer kontinuumsmechanischen Feldtheorie wäre die Änderung der inneren Energie eines materiellens Teilchens pro Zeiteinheit durch die materielle Ableitung von de gegeben, d.h.

$$\frac{De}{Dt}=\delta\dot{w}+\delta\dot{q}. \tag{4.34}$$

Diese Gleichung vergleichen wir jetzt mit der Gl. (4.33). Auf der linken Seite von Gl. (4.33) erkennen wir die materielle Ableitung von e, und durch leichtes Umschreiben dieser Gleichung erhalten wir

$$\frac{De}{Dt}=\rho^{-1}\sigma_{ij}\dot{\epsilon}_{ij}+(h-\rho^{-1}q_{i,i}). \tag{4.35}$$

Aus diesem Vergleich schliessen wir, dass der zweite und der dritte Term auf der rechten Seite (die Terme in der Klammer) zusammen die pro Zeit- und Masseneinheit dem materiellen Teilchen zugeführte Wärme darstellen. Aus dem Vergleich der Gl. (4.35) mit Gl. (4.34) wird ebenfalls klar, dass die Summe $\rho^{-1}\sigma_{ij}\dot{\epsilon}_{ij}$ der geleisteten Arbeit pro Zeit- und Masseneinheit entspricht, d.h.

$$\delta\dot{w}=\rho^{-1}\sigma_{ij}\dot{\epsilon}_{ij}.$$

Mit

$$\sigma_{ij}=\sigma_{kk}\delta_{ij}+\sigma_{ij}^{(d)}$$

erhalten wir

$$\rho\,\delta\dot{w}=\sigma_{kk}\dot{\epsilon}_{ii}+\sigma_{ij}^{(d)}\dot{\epsilon}_{ij}. \tag{4.36}$$

Es kann weiterhin noch gezeigt werden, dass der erste Term auf der rechten Seite dem reversiblen Anteil der Leistung entspricht. Der zweite Term ist die Leistung, die infolge der Deformation des

Körpers irreversibel in Wärme umgewandelt wird. Die irreversible Arbeit pro Zeit- und Volumeneinheit ist daher mit

$$\Phi := \dot{\epsilon}_{ij}\sigma_{ij}^{(d)}$$

durch die **Dissipationsfunktion** Φ gegeben.

Für inkompressible Medien, bei welchen $v_{i,i} = \dot{\epsilon}_{ii} = 0$ ist, verschwindet der erste Term auf der rechten Seite von Gl. (4.36), d.h.

$$\rho\,\delta\dot{w} = \sigma_{ij}^{(d)}\dot{\epsilon}_{ij}.$$

Demnach wird bei einem kriechenden (Beschleunigungsterme vernachlässigbar klein) inkompressiblen viskosen Medium die gesamte von den äusseren Kräften geleistete Arbeit im Körper in Wärme umgewandelt.

In einem Gletscher ist die dem Eis durch Strahlung zugeführte Wärme h in der Regel vernachlässigbar klein. Ausserdem ist Gletschereis inkompressibel ($v_{i,i} = \dot{\epsilon}_{ii} = 0$). Die Energiebilanz können wir daher in diesem Fall auch in der Form

$$\rho\frac{De}{Dt} = \Phi - q_{i,i} \tag{4.37}$$

schreiben. Die spezifische innere Energie eines dichtebeständigen Materials ändert sich nach der obigen Gleichung aus zwei Gründen: Erstens infolge der inneren Reibung, wodurch mechanische Energie irreversibel in Wärme dissipiert, und zweitens infolge von Wärmeleitung.

4.7 Bilanz der Entropie

Wird einem System die differentielle Wärmemenge δq reversibel zugeführt, ändert sich die Entropie (S) des Systems um den Betrag

$$dS = \frac{\delta q}{T} \qquad \text{(reversible Prozesse)}.$$

Nach dem zweiten Hauptsatz der Thermodynamik nimmt bei jedem irreversibel ablaufenden Prozess die Entropie zu, d.h

$$dS \geq \frac{\delta q}{T} \qquad \text{(alle in der Natur vorkommenden Prozesse)}.$$

Bei jedem Prozess kann also die Entropie nur zunehmen.

Für einen materiellen Körper wäre die entsprechende kontinuumsmechanische Formulierung des zweiten Hauptsatzes

$$\frac{DS}{Dt} \geq \frac{\delta\dot{q}}{T}. \tag{4.38}$$

Sowie die Änderung der gesamten Wärme eines materiellen Körpers sich aus der Summe der im Körper produzierten Wärme

$$\int_V \frac{\rho h}{T}\,dV \qquad \text{(Wärmeproduktion)}$$

und der dem Körper über den Rand zugeführten Wärme

$$-\int_A \frac{q_i n_i}{T}\,dA \qquad \text{(Wärmefluss)}$$

zusammensetzt, entspricht die Änderung der gesamten Entropie eines solchen Körpers der Summe der im Körper produzierten Entropie[3]

$$\int_V \frac{\rho h}{T}\,dV \qquad \text{(Entropieproduktion)}$$

[3]Diese Grösse wird in der deutschsprachigen Fachliteratur auch die *Entropiezufuhr* genannt. Der englische Fachausdruck ist *entropy source*.

und der dem Körper über den Rand zugeführte Entropie

$$-\int_A \frac{q_i n_i}{T}\, dA \qquad \text{(Entropiefluss)}.$$

Wenn wir mit s die Entropie pro Masseneinheit (die **spezifische Entropie**) bezeichnen, so erhalten wir Gl. (4.38) in der Form

$$\frac{D}{Dt}\int_V \rho s\, dS \geq -\int_A \frac{q_i n_i}{T}\, dA + \int_V \frac{\rho h}{T}\, dV. \tag{4.39}$$

Um die lokale Form der Entropiebilanz zu erhalten, formen wir die linke Seite der obigen Gleichung mit der Hilfe der Gl. (4.16) um. Der Satz von Gauss wird dann benutzt, um das Oberflächenintegral auf der rechten Seite in ein Volumenintegral umzuwandeln. Daraus folgt, dass

$$\int_V \left(\rho\frac{Ds}{Dt} - \frac{\rho h}{T} + \left(\frac{q_i}{T}\right)_{,i}\right) dV \geq 0.$$

Da dieses Integral für jedes Volumen verschwinden muss, gilt

$$\rho\frac{Ds}{Dt} - \rho\frac{h}{T} + \left(\frac{q_i}{T}\right)_{,i} \geq 0. \tag{4.40}$$

Diese Ungleichung ist als die **Clausius-Duhem-Ungleichung** bekannt.

Jetzt ist aber

$$\left(\frac{q_i}{T}\right)_{,i} = \frac{1}{T}q_{i,i} - \frac{1}{T^2}\, q_i T_{,i},$$

und nach Gl. (4.33) (Energieerhaltung) gilt

$$q_{i,i} = \sigma_{ij}\dot{\epsilon}_{ij} + \rho h - \rho\frac{De}{Dt}.$$

Somit lässt sich Gl. (4.37) auch in der Form

$$\rho T\frac{Ds}{Dt} + \sigma_{ij}\dot{\epsilon}_{ij} - \rho\frac{De}{Dt} - \frac{1}{T}\, q_i T_{,i} \geq 0 \tag{4.41}$$

schreiben.

Der Nutzen der oben aufgeführten Ungleichungen besteht vor allem darin, dass sie gewisse Einschränkungen für die möglichen Formen von Materialgesetzen liefern. So muss z.B. für jedes Fluid der funktionale Zusammenhang zwischen den Dehnungsraten $\dot{\epsilon}_{ij}$ und den Komponenten des Spannungstensors σ_{ij}, wie auch der funktionale Zusammenhang zwischen der Temperatur und dem Wärmestromsvektor dergestalt sein, dass die Ungleichung (4.37) für jede mögliche Lösung der Feldgleichungen (Masse, Impuls, Energie) erfüllt ist.

Chapter 5

Constitutive equations

5.1 Introduction

The field equations can be summarized as follows

$$v_{i,i} = 0 \quad \text{(mass)} \tag{5.1}$$

$$\sigma_{ki,k} + \rho b_i = 0 \quad \text{(linear momentum)} \tag{5.2}$$

$$\sigma_{ij} - \sigma_{ji} = 0 \quad \text{(angular momentum)} \tag{5.3}$$

$$\rho \frac{De}{Dt} - \dot{\epsilon}_{ij}\sigma^{(d)}_{ij} + q_{i,i} - \rho h = 0 \quad \text{(energy).} \tag{5.4}$$

The mass-conservation equation only holds for an incompressible continuum, and in the linera-momentum conservation equation the acceleration term has been ignored. These simplifications can almost always be made in glacier mechanics.

In the system above there are eight scalar equations (mass 1, linear momentum 3, angular momentum 3, energy 1) and 16 unknown fields (v_i (3), σ_{ij} (9), q_i (3), e (1)). The system is, thus, not closed, that is using these equations alone it is not possible to determine all the field functions. We should, of course, not expect this to be possible. Note that the field equations listed above hold equally well for all incompressible materials (if acceleration terms can be ignored) and do not account for differences in the thermal and the mechanical properties of materials. We have yet to give a description of how the material deforms in the presence of applied stresses, and this description is needed to close the system.

The mechanical and thermal properties of the material under study are described by the so called *constitutive equations* (German: *Konstitutivgleichungen* or *Materialgleichungen*). Usually a set of three equations: a law of heat conduction, a caloric equation of state, and a material law, are needed to completely specify all mechanical and thermal properties. The law of heat conduction describes the relation between the heat-flux vector and the temperature gradient. A caloric equation of state gives the connection between the specific internal energy and the temperature. A material law relates the stress tensor and the other field variables.

5.2 Stokesian fluid

A constitutive equation expresses the physical properties of the material concerned and can not be derived using theoretical arguments alone. Nevertheless, theoretical arguments can be used to narrow down the possible mathematical form of the constitutive equation.

One can, for example, show that if the stress tensor is a function of the rate-of-deformation tensor, and both tensors possess the same symmetries, then the most general form of the material law is

$$\sigma_{ij} = \alpha\,\delta_{ij} + \beta\dot{\epsilon}_{ij} + \gamma\dot{\epsilon}_{ik}\dot{\epsilon}_{kj} \tag{5.5}$$

where α, β, and γ are some scalar functions of density, temperature, and the three invariants of the stretching tensor.

Eq. (5.5) represents the most general relationship between stresses and strain-rates under fairly general assumptions. These assumptions are that the fluid is isotropic and homogeneous, with the stress tensor being a continuous function of the rate-of-deformation tensor and of some thermodynamical variables only. We will not proof in detail that Eq. (5.5) follows from these assumptions. We note, however, that the assumption of isotropic fluid implies that the functional relationship

$$\sigma = \mathbf{f}(\mathbf{D})$$

between the stress tensor and the rate-of-deformation tensor must fulfill

$$\mathbf{f}(\mathbf{Q}\mathbf{D}\mathbf{Q}^T) = \mathbf{Q}\mathbf{f}(\mathbf{D})\mathbf{Q}^T, \tag{5.6}$$

where $\mathbf{Q}$ is a rotation matrix. This implies that $\mathbf{f}$ is an isotropic tensor function.[1] All isotropic tensor functions of second order tensors can be written on the form

$$\mathbf{f}(\mathbf{D}) = \alpha\mathbf{1} + \beta\mathbf{D} + \gamma\mathbf{D}^2$$

where α, β, and γ are scalar functions, from which Eq. (5.5) follows.

If we make the additional assumption that in the absence of deformation the stress is hydrostatic (sometimes used as a definition of fluids) then α in (5.5) must be equal to the negative of the pressure p, that is $\alpha = -p$. A fluid having constitutive equations on this form is called *Stokesian Fluid* or sometimes *Reiner-Rivilin Fluid*.

5.3 Incompressible fluid

For a compressible material a change in pressure will in general lead to changes in density. In this case there is a functional relationship between pressure (p) and density (ρ), and the *thermodynamical pressure* p is given by the *kinetic equation of state*

$$p = p(\rho, T).$$

For an incompressible material, however, there is no change in density with pressure, and thus no corresponding relationship between these two variables. The pressure can therefore not be derived from density and temperature, and becomes a new independent dynamical variable. In the absence of a kinetic equation of state defining the (thermodynamical) pressure we must look for alternative ways of defining the pressure in some sensible way.

In a fluid at rest all the normal stresses are equal and all the shear stresses are zero. That is

$$\sigma_{ij} = -\bar{p}\,\delta_{ij},$$

whenever $\dot{\epsilon}_{ij} = 0$, with $\bar{p}$ defined as

$$\bar{p} := -\frac{1}{3}\sigma_{ii}.$$

The field variable $\bar{p}$ is called the *mechanical pressure*. Note that (5.3) is a definition, and that the mechanical pressure is simply the negative of the mean value of the normal stresses. [2] In what follows we will be dealing with incompressible materials only, and for this reason we will simply write p for the mechanical pressure.

[1] A tensor whose components remain unchanged under all coordinate transformations is called an isotropic tensor.

[2] It can be shown that for a compressible material

$$p - \bar{p} = -\left(\lambda + \frac{2}{3}\right)\frac{1}{\rho}\frac{D\rho}{Dt}.$$

Taking the limit $D\rho/Dt \to 0$ we find $p = \bar{p}$. This shows that for a incompressible material the mechanical pressure is the correct limit of the thermodynamical pressure as defined for a compressible material.

With $\alpha = -p$ the general form of the constitutive law can now be written as

$$\sigma_{ij} = -p\,\delta_{ij} + \beta\dot{\epsilon}_{ij} + \gamma\dot{\epsilon}_{ik}\dot{\epsilon}_{kj}. \tag{5.7}$$

Using the definition of mechanical pressure and the definition of deviatoric stresses ($\sigma_{ij}^{(d)} = \sigma_{ij} - p\,\delta_{ij}$), this expression can also be written as

$$\sigma_{ij}^{(d)} = \beta\dot{\epsilon}_{ij} + \gamma\dot{\epsilon}_{ik}\dot{\epsilon}_{kj}. \tag{5.8}$$

Because changes in pressure do not lead to any deformations in an incompressible material it is usually much more convenient to work with Eq. (5.8) than Eq. (5.7). The constants β and γ can be functions of thermodynamical variables and the three invariants of the stretching tensor.

Instead of writing the deviatoric stresses as a function of strain rates as done in (5.8) one can equally well invert this relationship (not always in a closed form) and write the strain-rates as functions of the deviatoric stresses.

5.4 The constitutive law of ice

Considerable experimental work has been done towards determining the constitutive law of ice. These experiments suffer from the fact that it is, for practical reasons, not possible to conduct mechanical tests on ice at strain-rates that are typically found in glaciers and ice sheets (10^{-4} to $10^{-2}\,\mathrm{a}^{-1}$). Strain rates used in the laboratory are at least two orders of magnitude larger than those typically found in nature. If one does not agree with the claim that it is possible to extrapolate the experimental findings to regions of strain-rates and stresses typically found in glaciers, it must concluded that none of the laboratory work done to date on the rheology of ice has any relevance to the rheology of glaciers!

Ice comes in various crystalline forms. There are at least twelve states of ice, with the twelfth structure being found in the late nineties. At normal pressures and temperatures there is only one stable crystal structure of ice, Ice Ih (h for hexagonal). In Ice Ih each oxygen atom is bounded to two hydrogen atoms by electron orbitals, forming a covalently bounded H_2O molecule. The protons of the two hydrogen nuclei are not completely shielded by the electrons shared with the oxygen atom, leaving the hydrogens positively charged. There are further two bounding orbitals with two electrons each (two so called lone pairs). These four orbitals radiate tetrahedrally from the oxygen nucleus, forming an almost exact tetrahedral arrangement. Two of these orbitals are associated with excessive positive charge and the other two with excessive negative charge. Each H_2O can, therefore, hock up with four other H_2O molecules. In Ice Ih this gives rise to a layered structure of H_2O molecules. Within the layers the molecules are arranged in hexagonal structures. The plane parallel to the layer-structure is called the *basal plane*. The direction perpendicular to the basal plane is referred to as the *c-axis*.

From laboratory experiments it is known that ice deforms much more easily in the basal plane than along any other planes (or slip systems). The mechanical properties of single ice crystals are therefore strongly anisotropic. If, however, the individual ice crystals are randomly oriented (random fabric), the bulk mechanical properties of glacier ice may nevertheless be described through an isotropic constitutive relationship on the form (5.8). After prolonged constant shearing individual ice crystals, however, tend to become reoriented and reorganized through a number of processes collectively referred to as *dynamic recrystallization*, and with time this can lead to a strongly anisotropic ice. On the other hand, for ice warmer than about -10 degrees Celsius, the current understanding is that enough planes of deformation are active for ice to remain isotropic indefinitely even when subjected to prolonged periods of constant deformation.

The ability of ice, as of all crystalline solids, to deform depends critically on the density and the types of various defects in the crystal structure. Such packing irregularities can be numerous. In metals a density of 10^{12} per square centimeter is not uncommon. Packing irregularities within a crystal lattice are of various types. They can be categorized in terms of their dimensionality. Examples are *point defects*, where atoms are missing or some irregularities are found at isolated

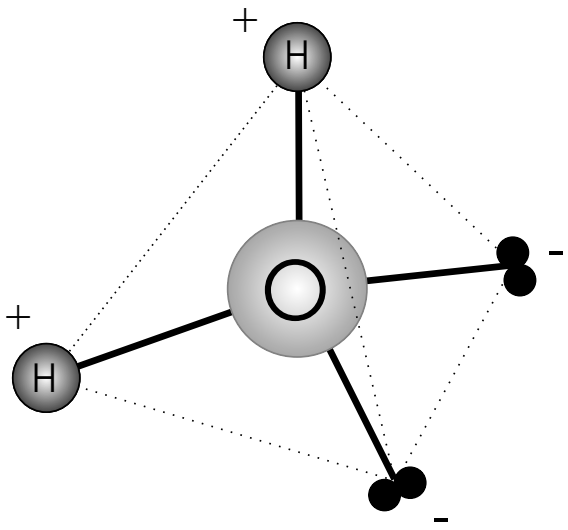

Figure 5.1: H_2O molecule. The four corners of the tetrahedral are formed by the hydrogen atoms and the two 'lone pair' electrons (black), with the oxygen atom in the center. There are excessive positive and negative charges associated with the hydrogen atoms and the lone pairs, respectively.

locations within the crystal lattice, and *linear defects* where groups of atoms or molecules have been displaced, are missing or have been introduced, with the result than a line of defects within the lattice is formed, and *plane defects*. Each of these different groups of defects come in various types. Examples of point defects are: lattice vacancies, substitutional impurities, interstitial impurities, and proper intersitials. Line defects include screw and edge dislocations, and plane defects include small-angle-grain boundaries, stacking faults, and twin boundaries.

Dislocations are linear defects. Dislocations are movable and the stress needed to move a dislocation is, in general, much lower than the stress needed to deform a perfect crystalline. Dislocations reduce the strength of a material well below its theoretical strength as expected on the basis of the strength of individual atomic bounds within the crystal lattice. Dislocation is a key factor affecting the mechanical properties of solids. Insight into the possible rheology of materials can be gained from knowledge of the dynamics of dislocations.

For a given stress (σ) the strain-rate within a sample of ice can be expected to be given by the product of the density of dislocations, the rate at which they move through the material, and the displacement associated with each dislocation, that is

$$\dot{\epsilon} = \rho_d v_d \phi b \tag{5.9}$$

where ρ_d is the density (number of dislocation lines that cut a cross-sectional area) of mobile dislocations, v_d is the velocity by which dislocation travel through the crystal lattice, ϕ is an orientation factor related to the orientation of the slip plane with respect to applied stress, and b the Burges vector (together with the orientation factor the atomic displacement associated with the dislocation.) Eq. (5.9) is known as the *Orowan relation*.

The dislocation density reflects a balance between the creation and destruction of dislocations. It is, in general, difficult to derive the density as a function of stress. One possibility is equating the externally applied stress on an ice crystal with the internal stress associated with the dislocations within it as follows. The stress associated with a dislocation as it gets activated will be proportional to the displacement (Burges vector) times some material constant μ which we can think of as the shear modulus or as a spring constant. The stress associated with each dislocation will go up with with increasing distance d between dislocations (there are less dislocations to take up the stress),

which means that less applied stress is needed to activate individual dislocation

$$\sigma \propto \frac{b\mu}{d}.$$

The density of dislocations is measured as number of dislocations per area. The average distance d between dislocations is therefore proportional to $\rho_d^{-1/2}$. The internal stress is therefore

$$\sigma \propto b\mu\rho_d^{1/2},$$

or

$$\rho_d \propto \frac{\sigma^2}{b^2\mu^2}, \tag{5.10}$$

showing that when the generation rate of dislocations equals the annihilation rate, the dislocation density is proportional to the square of the applied stress. Combining (5.10) with the Orowan relation (5.9) gives

$$\dot{\epsilon} \propto \frac{v\phi\sigma^2}{b\mu^2} \tag{5.11}$$

The dislocation velocity (v) is known both theoretically and experimentally to be linearly proportional to the applied stress. (At -18 degrees Celsius v in ice is, for example, equal to $2\,\mu\mathrm{m\,s^{-1}\,MPa^{-1}}$.) It is, furthermore, strongly dependent on temperature, with

$$v \propto \sigma e^{-\frac{Q}{kT}} \tag{5.12}$$

where Q is an activation energy, k the gas constant, and T is temperature. Inserting Eq. (5.12) into (5.11) leads to

$$\dot{\epsilon} \propto \frac{\phi}{b\mu^2}\,\sigma^3\,e^{-\frac{Q}{kT}} \tag{5.13}$$

suggesting a power-law for the rheology of ice with an exponent of 3.

There are a number of reasons why (5.13) may not be correct for polycrystalline ice. Grain boundaries can, for example, immobilize dislocations thereby reducing the number of active dislocations. But grain boundaries can also act as sources for dislocations. Differently oriented grains may have also difficulties deforming in a mutually compatible way leading to non-uniform distribution of stresses with the polycrystalline ice. As a consequence, the grains may slide with respect to each other, or the grain boundaries may migrate. Furthermore, recrystallization may occur with new grains being formed with slip planes better oriented for deformation, or existing grains may become subdivided (polygonization).

In deriving Eq. (5.13) use was made of (5.10) which predicts the dislocation density, ρ_d, to increase with σ. If, for some reason, ρ_d where to remain constant with increasing stress, the flow law would become linear. There is some indication of this happening in ice at small stresses (less than 0.01 MPa).

5.5 Creep curves

Creep is here defined as a time-dependent inelastic deformation at constant stress. When uniaxial compression is applied to a sample of polycrystalline ice, the resulting curve describing strain as function of time can be divided into a number of regions. Initially, there is an instantaneous elastic response. This is follows by a combination of *delayed elastic strain*[3] and *viscous strain*. The delayed elastic strain is fully recoverable following an unloading of the ice. The viscous strain, on the other hand, not. With time, the contribution of delayed elastic strain to the creep-rate (the slope of the strain versus time curve) decreases. The minimum value in observed slope is

[3]Other terms used for delayed elastic strain are primary creep, transient creep, recoverable creep and pseudo-elastic strain

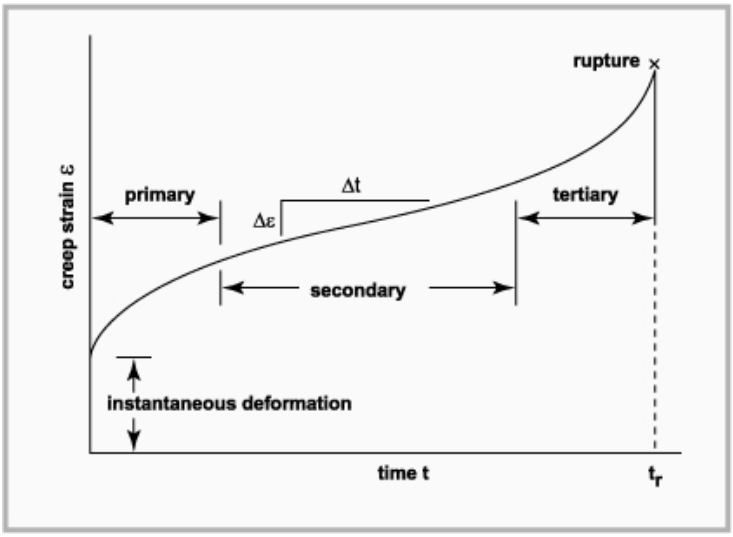

Figure 5.2:

referred to as the *secondary creep*. In laboratory experiments secondary creep is usually reached after about 1% strain, but this is expected to depend on the contribution of delayed elastic strain to the total strain and therefore on the applied stress level. In laboratory experiments, secondary creep is not a steady-state process and is eventually followed by a renewed increase in creep-rate, with is referred to a *tertiary creep*.

The delayed elastic strain is of little interest with respect to the long-term deformation of glacier ice. Tertiary creep seems to be at least partially associated with internal crack formation and dynamic recrystallization (Duval et al., 1983), and ultimately may simply be a consequence of delayed elastic strain. This indicates that as the applied stress is reduced to typical values of deviatoric stresses in glaciers (0.1 MPa) no tertiary creep will be observed. It is therefore the secondary creep, or the minimum value of the slope of the creep versus time curve, which is of relevance with regard to the rheology of glacier ice at stresses typically found in glaciers.

Laboratory studies of isotropic polycrystalline ice under uniaxial compressive and tensile loading conducted in the early fifties showed that for secondary creep the strain rate is related to stress through a power law on the form

$$\dot{\epsilon} = A\,\sigma^n \tag{5.14}$$

where $\dot{\epsilon}$ is the strain-rate along the same axis as the applied stress σ Steinemann (1954, 1958a,b); Glen (1955) The exponent n is thought to be in the range form 2 to 5. At strain rates below $10^{-5}\,\mathrm{s}^{-1}$ and stresses below 1 MPa, many authors have reported that n close to 3 gives the best fit to the data (e.g. Glen, 1952; Steinemann, 1958a; Mellor and Cole, 1983; Mellor and Smith, 1967) although some conflicting reports are found in the literature. Theoretical studies of dislocation dynamics in ice suggest that the $n = 3$ results from dislocation glide along the basal plane. Dislocations can also climb from one glide plane to another within a grain. Eventually, dislocations hit grain boundaries and continuous deformation requires changes in crystal structure. For stresses above 1 MPa an exponent of $n = 4$ seems to give a better fit Steinemann (1954); B. et al. (1992). This suggest dislocation glide along some of the hard slip systems.

A power-law relationship between stress and strain-rates is not only found for ice, but for numerous other materials and substances including metals close to their melting point (and ketchup too). Note that almost all test have been performed on cold ice and the rheology of temperate ice is much less well constrained through laboratory experiments.

As mentioned above, it is difficult, if not impossible, to perform laboratory experiments for stresses and strain-rates typical of glaciers and ice sheets. The $n = 3$ exponents, for example, was obtained in experiments where the strain-rates were at least two order of magnitudes larger than

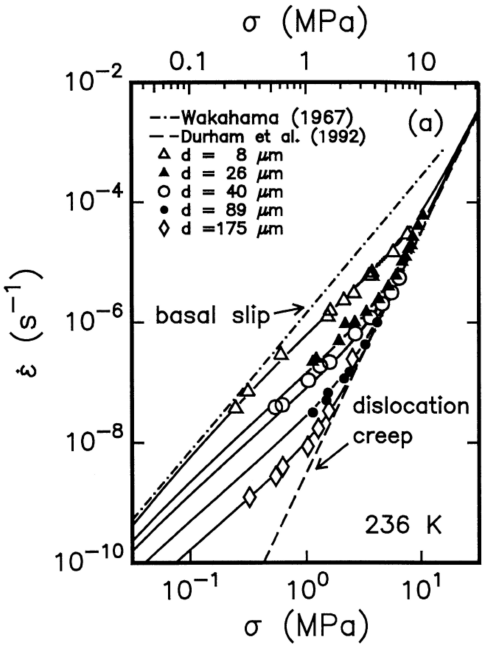

Figure 5.3: Stress and strain-rate relationship of fine-grained ice as determined by Goldsby and Kohlstedt (2001). The authors identify three creep regimes. At highest stresses the flow law exponent is $n = 4$, for intermediate stresses $n = 1.8$, and $n = 2.4$ at the lowest stresses.

typical of glaciers. Recently Goldsby and Kohlstedt (2001) have conducted series of experiments of ice at temperatures between 170 and 268 K, differential stresses of 0.2 to 20 MPa with strain rates varying from 10^{-8} to 10^{-4} per sec. One of the novel aspects of their experiments is the use of ice samples of various grain sizes, including very fine-grained samples with sizes from 3 to 100 μm. Goldsby and Kohlstedt (2001) conclude that the flow of ice is controlled by at least four different deformation mechanisms, each one being characterized by different stress exponents n and activation energy Q. At high stresses, above 1 MPa, $n = 4$ and the primary deformation is dislocation creep along both the basal and the nonbasal slip systems. At somewhat lower stresses, a combination of both grain boundary sliding and basal slip determine the flow.[4] At first the GBS is rate-controlling mechanism deformation mechanism, and $n = 1.8$. As the stresses are lowered further, GBS becomes faster and the basal slip is now the rate-controlling mechanism. This gives rise to a stress exponent $n = 2.4$. At even lower stresses a different process, diffusion of grain boundaries, starts to dominate the deformation of ice. For this diffusional flow $n = 1$. Typical deviatoric stresses in glaciers and ice sheets are either too small or too large for both the dislocation creep regime ($n = 4$) and the diffusional flow regime with ($n = 1$). The $n = 2.4$ regime (basal slip accommodated by grain boundary sliding) is also only of relevance for temperatures close zero and deviatoric stresses smaller than 0.01 MPa. According to Goldsby and Kohlstedt (2001) it is the $n = 1.8$ regime (grain boundary sliding accommodated by basal slip) which is the rate-limiting creep mechanism over most glacilogically significant stresses and temperatures.

Previously secondary creep in ice was thought to be independent of grain size, however Goldsby and Kohlstedt (2001) conclude that for the $n = 1.8$ regime strain rates do depend on grain size. This grain-size dependency can be described as a power law with a grain size exponent of -1.4.

Goldsby and Kohlstedt (2001) propose a new constitutive equation for polycrystalline ice on the form

$$\dot{\epsilon} = \dot{\epsilon}_{\text{diff}} + \left(\dot{\epsilon}_{\text{basal}}^{-1} + \dot{\epsilon}_{\text{gbs}}^{-1}\right)^{-1} + \dot{\epsilon}_{\text{disl}} \tag{5.15}$$

[4] In *Grain boundary sliding* the grains slide past each other along their common boundary. Grain boundary sliding (GBS) requires some accommodating process (for example basal slip) for it to be sustained.

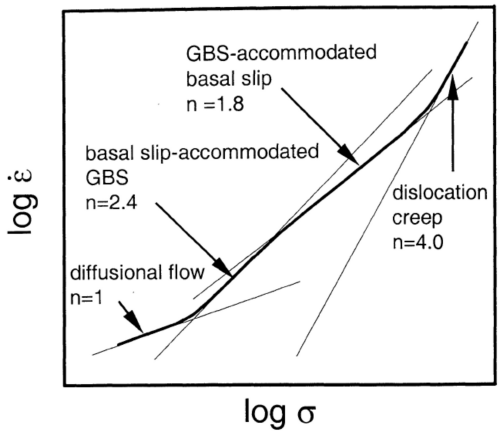

Figure 5.4: Schematic diagram showing the dominating creep mechanism for ice at different stresses. From Goldsby and Kohlstedt (2001).

where each term on the right-hand side has the form

$$\dot{\epsilon} = A\sigma^n d^{-p} e^{-(Q+PV)/RT},$$

where A is a rate factor, d is grain size, Q the activation energy, V the activation volume, R the gas constant, and n and p stress and grain size exponents. In Eq. (5.15) the subscripts refer to diffusional flow (diff) with $n = 1$, basal slip accommodated by grain boundary sliding (basal), grain boundary sliding accommodated by basal slip (gbs), and dislocation creep (disl).

5.6 Flow law of ice

The experimental findings on the rheology of ice described above were made using fairly simple stress configurations (simple shear, uniaxial compression, etc.). They can be generalized for three-dimensional stress, but only by making some additional assumptions such as that γ in Eq. (5.8) is zero. In glaciology, the constitutive equation is usually written in an inverse form as compared to Eq. (5.8), that is the strain rates are expressed as functions of the deviatoric stresses

$$\dot{\epsilon}_{ij} = f\sigma_{ij}^{(d)}.$$

Here f is some scalar function of temperature and the invariants of the deviatoric stress tensor

$$f = f(T, I_{\boldsymbol{\sigma}^{(d)}}, II_{\boldsymbol{\sigma}^{(d)}}, III_{\boldsymbol{\sigma}^{(d)}}).$$

For incompressible materials $I_{\boldsymbol{\sigma}^{(d)}} = 0$. It is also generally thought that the value of the third invariant does not affect the deformation of ice, that is

$$f = f(T, II_{\boldsymbol{\sigma}^{(d)}})$$

although this assumption has sometimes been questioned.

Using these assumption and generalizing the power law (5.14) we then find

$$f = A(T')\,\tau^{n-1},$$

where

$$\begin{aligned} \tau &:= \sqrt{-II_{\boldsymbol{\sigma}^{(d)}}} \\ &= \sqrt{\frac{1}{2}\sigma_{ij}^{(d)}\sigma_{ji}^{(d)}} \qquad \text{(Eq. (3.19))}. \end{aligned}$$

This "law" is in glaciology known as Glen's flow law. It has been used extensively in glaciological modeling work. To call it a "law" is misleading. It is not a law in the same sense as, for example, the conservation of mass is a law of nature. The value of the flow parameter n is thought to be within the range 2 to 5. For some unknown reasons, most modeling work seems to be done using $n = 3$, but there does not seem to be any strong justification for using this particular value of n, neither from field work or laboratory experiments.

The value of A depends strongly on the homological temperature

$$T' = T - T_M$$

where T_M is the pressure-dependent melting temperature of ice. Glen's flow law can be written on the form

$$\dot{\epsilon}_{ij} = A(T')\,\tau^{n-1}\sigma^{(d)}_{ij}. \tag{5.16}$$

Inverting Glen's flow law

The form of the Glen's flow law given by Eq. (5.16) can be used to determine strain rates for given deviatoric stress tensor. If we want to calculate deviatoric stresses as functions of strain rates this expression must be inverted.

Starting with

$$\dot{\epsilon}_{ij} = A\,\tau^{n-1}\,\sigma^{(d)}_{ij}$$

we multiply both sides with $\dot{\epsilon}_{kl}$,

$$\dot{\epsilon}_{kl}\dot{\epsilon}_{ij} = A\,\tau^{n-1}\,\sigma^{(d)}_{ij}\dot{\epsilon}_{kl}$$

using the Glen's flow law on the right hand side

$$\dot{\epsilon}_{kl}\dot{\epsilon}_{ij} = A\,\tau^{n-1}\,\sigma^{(d)}_{ij}\,A\,\tau^{n-1}\,\sigma^{(d)}_{kl}.$$

By now multiplying both sides with $\delta_{ik}\delta_{lj}$

$$\delta_{ik}\delta_{lj}\dot{\epsilon}_{kl}\dot{\epsilon}_{ij} = A^2\,\tau^{2(n-1)}\,\sigma^{(d)}_{ij}\sigma^{(d)}_{kl}\delta_{ik}\delta_{lj}$$

we get the sums

$$\underbrace{\frac{1}{2}\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}}_{II_{\dot{\epsilon}}} = A^2\tau^{2(n-1)}\ \underbrace{\frac{1}{2}\sigma^{(d)}_{ij}\sigma^{(d)}_{ij}}_{\tau^2},$$

and therefore

$$II_{\dot{\epsilon}} = A^2\tau^{2\,n},$$

or

$$\dot{\epsilon} = A\,\tau^n. \tag{5.17}$$

after having defined $\dot{\epsilon}$ as

$$\dot{\epsilon} := \sqrt{\frac{1}{2}\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}}.$$

The quantity $(\dot{\epsilon})$ is the *effective strain rate*, in analogy with the quantity τ which is the *effective stress*. Eq. (5.17) neatly expresses the relationship between the effective stress and the effective strain rate.

According to (5.16) we have

$$\sigma^{(d)}_{ij} = A^{-1}\tau^{1-n}\,\dot{\epsilon}_{ij}.$$

By inserting Eq. (5.17) in this equation we obtain

$$\begin{aligned}\sigma^{(d)}_{ij} &= A^{-1}A^{\frac{n-1}{n}}\,\dot{\epsilon}^{\frac{1-n}{n}}\,\dot{\epsilon}_{ij}\\ &= A^{-1/n}\dot{\epsilon}^{\frac{1-n}{n}}\,\dot{\epsilon}_{ij},\end{aligned}$$

and then

$$\boxed{\sigma_{ij}^{(d)} = A^{-1/n}\, \dot{\epsilon}^{\frac{1-n}{n}} \dot{\epsilon}_{ij}.} \tag{5.18}$$

This is the expression that we were looking for, giving deviatoric stresses as a function of strain rates.

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